Use It or Lose It: Efficiency and Redistributional Effects from Wealth Taxation Computational Appendix

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1 Solution method

We focus on the stationary equilibrium of the model. We first make a guess for the the stationary distribution of agents and aggregate variables. Then, we solve for the policy functions of the individual agents by backward induction (using the endogenous grid method, henceforth EGM). We use these policy functions to update the distribution (using Young (2010)'s histogram method). The updated distribution implies new aggregate variables. We repeat this process until we achieve convergence of the distribution of agents (and hence of the aggregate variables and policy functions).

1.1 Initialization

To initialize the algorithm one needs to construct grids for all the state variables (a, z^p, κ, e, z^s) and transition matrices for the exogenous ones. It is also necessary to determine age distribution and survival probabilities along with the distribution of efficiency units by age and the replacement ratio ϕ for retirement.

First, consider the grids for the state variables. The exogenous states (z, κ, e) are discretized using Tauchen (1986)'s method, obtaining both the grids and the transition matrices P_z, P_κ and P_e that imply conditional CDFs and stationary distributions denoted by G_z, G_κ and G_e. In particular we assume that the variables follow the processes:

$$\begin{aligned}\ln z^{p'} =& \rho_z \ln z^p + \sigma_z \epsilon_z;\\ \ln \kappa' =& \rho_\kappa \ln \kappa + \sigma_\kappa \epsilon_\kappa;\\ \ln e' =& \rho_e \ln e + \sigma_e \epsilon_e;\end{aligned}$$

where $\epsilon_i \sim N(0,1)$ for $i = \{z^p, \kappa, e\}$. Tauchen's method requires the specification of the number of nodes and the dispersion of the grid measured by number of standard deviations around the mean. In our parametrization both κ and e are discretized using 5 nodes with a dispersion of 3 standard deviations. z^p is discretized using 11 nodes with a dispersion of 5 standard deviations. The larger dispersion for the z^p grid adds nodes that characterize agents with large (but rare) productivity levels. This comes at the cost of more grid nodes, yet, this cost can be partially avoided by merging the lower end of the grid into a single node. There is little cost of doing this in our model because all the agents with low values of z^p behave similarly, they are not productive enough to operate their own firm at a large scale and prefer to lend most of their assets to more productive agents at the market rate. Moreover, they do not care about the transition probabilities of z^p because they have no direct preferences for their offspring. In our parametrization we merge the lowest 3 nodes of the z^p grid, leaving a grid of 9 nodes. The transition matrix is adjusted and the node is given the value of the average of the nodes being merged (the average is taken using the stationary distribution of the original grid G_z). The mass of agents in the merged nodes is 0.62% .

It is convenient for us to compute distribution of e by age, which is possible because all agents are born with $e = 1$, which corresponds to the $\frac{n_e+1}{2}^{th}$ node of the grid (with n_e an odd number). We denote this distribution by $G_e(h, i_e)$, where h is the age and i_e is the index of e in the grid. For age equal 1 we have :

$$G_e^{age}\left(1, \frac{n_e+1}{2}\right) = 1 \qquad G_e^h(1, i_e) = 0 \quad \forall_{i_e \neq \frac{n_e+1}{2}}.$$

Then we have for $h \geq 2$:

$$G_e^{age}(h, i_e) = \sum_{k=1}^{n_e} G_e^h(h-1, k) P_e(k, i_e).$$

The stochastic component of the agent's productivity follows a three state first order Markov process where $z^s \in \{H, L, 0\}$ and the transition matrix is given by

$$\Pi_z = \begin{bmatrix} 1 - p_1 - p_2 & p_1 & p_2 \\ 0 & 1 - p_2 & p_2 \\ 0 & 0 & 1 \end{bmatrix}.$$

We assume that individuals whose permanents ability $z^p > z_{med} = 1$ start life in state $z^s = H$ and the rest starts at state $z^s = L$. We further assume that $p_1 = 0.05$ and $p_2 = 0.03$.

The effective productivity of the agent depends on the permanent and stochastic components of productivity following:

$$z = f(z^p, z^s) = \begin{cases} (z^p)^\lambda & \text{if } z^s = H \qquad \text{where } \lambda > 1 \\ z^p & \text{if } z^s = L \\ z_{min} & \text{if } z^s = 0 \end{cases},$$

where we set $\lambda = 1.5$ and $z_{min} = 0$.

The grid for assets $\vec{a}$ has n_a nodes between a_{min} and a_{max}. The grid is constructed as follows:

$$a_i = a_{min} + (i-1)^{\gamma_a} \frac{a_{max} - a_{min}}{n_a - 1},$$

where:

$$n_a = 201 \qquad a_{min} = 0.0001 \qquad a_{max} = 500000 \qquad \gamma_a = 4.$$

The maximum age and retirement age are $H = 81$ and $\overline{h} = 45$ respectively, taking 20 as age 1 in the model. Survival probabilities are constructed using population numbers from Bell and Miller (2002). Let pop_h be the population alive at age h, then:

$$s_h = \frac{\text{pop}_h}{\text{pop}_{h-1}}.$$

When an agent retires at age $\overline{h}$ she starts receiving social security income $y^R(\kappa, e_{\overline{h}})$ that depends on her type κ and the value of e at retirement in the following way:

$$y^R(\kappa, e) = \Phi(\kappa, e_{\overline{h}})\,\overline{E},$$

where Φ is the agent's replacement ratio, a function that depends on the agent's type and last transitory shock to labor productivity, and $\overline{E}$ corresponds to average earnings in the economy, given by: $\overline{E} = \frac{W\overline{N}}{I_1^{\overline{h}}}$, where W is the economy wide wage, $I_1^{\overline{h}} \leq 1$ is the measure of agents in working age, and $\overline{N}$ is the total number of effective hours worked in the economy:

$$\overline{N} = \sum_{i=1}^{\overline{h}-1} \int \hat{y}_h(\kappa, e)\, n_h(s)\, d\Gamma_h(ds).$$

The replacement ratio is progressive and satisfies:

$$\Phi(\kappa, e) = \begin{cases} 0.9\frac{y(\kappa,e)}{\overline{y}} & \text{if } \frac{y(\kappa,e)}{\overline{y}} \leq 0.3 \\ 0.27 + 0.32\left(\frac{y(\kappa,e)}{\overline{y}} - 0.3\right) & \text{if } 0.3 < \frac{y(\kappa,e)}{\overline{y}} \leq 2 \\ 0.91 + 0.15\left(\frac{y(\kappa,e)}{\overline{y}} - 2\right) & \text{if } 2 < \frac{y(\kappa,e)}{\overline{y}} \leq 4.1 \\ 1.1 & \text{if } 4.1 < \frac{y(\kappa,e)}{\overline{y}} \end{cases},$$

where $y(\kappa, e)$ is the average efficiency units that an agent of type κ gets conditional on having a given $e_{\overline{h}} = e$.

$$y(\kappa, e_{\overline{h}}) = \frac{1}{\overline{h}} \sum_{h=1}^{\overline{h}-1} \int \hat{y}_h(\kappa, e)\, d\Gamma_h(da, dz, \kappa, de),$$

the integral is taken with respect to the stationary distribution of agents by age and is taken over all possible asset holdings, types z, and histories of e such that $e_{\overline{h}}$ is the one given in the left hand side. Finally $\overline{y}$ is the average of $y(\kappa, e)$ across κ and e.

The value of $\overline{E}$ is determined in equilibrium, since it depends on the agent's decisions and the equilibrium wage, but the value of Φ can be obtained immediately since it only depends on the efficiency units and not on the labor supply decision of the household. Moreover, even though its definition involves the stationary distribution, only the distribution over age, κ and e is relevant for the calculations.

The values of $y(\kappa, e)$ and $\overline{y}$ are obtained through simulation using the following algorithm:

1. Choose a cohort size for the simulation. In our calibration we use 10 million agents.
2. For each agent draw a permanent labor productivity type κ from the stationary distribution G_κ.
3. Start each agent with $e_1 = 1$ and iterate forward in life from ages 2 to $\overline{h} - 1$ drawing the realizations of e from the transition matrix P_e.
4. For each agent and at each age (h) compute the efficiency units of labor given by κ and e:
$$\hat{y}_h(\kappa, e) = \kappa_h \kappa e \qquad \kappa_h = e^{\frac{60(h-1)-(h-1)^2}{1800}}.$$
5. Compute the average efficiency for each combination of $\left(\kappa, e_{\overline{h}-1}\right)$, with that $y\left(\kappa, e_{\overline{h}-1}\right)$ is obtained, and for all the agents, which yields $\overline{y}$.
6. Use the definition of Φ and the values of $y\left(\kappa, e_{\overline{h}-1}\right)$ and $\overline{y}$ to complete the procedure.

See subroutine "LIFETIME_Y_ESTIMATE" for more details.

1.2 Policy functions

We obtain the policy functions using the endogenous grid method and backward induction, given prices (r, w) and the evolution of exogenous variables. Let $s = (a, z^p, \kappa, e, z^s)$ be the state vector of an agent of age h. The utility function is

$$U(c, n) = \frac{\left(c^\gamma (1-n)^{1-\gamma}\right)^{1-\sigma}}{1-\sigma},$$

which implies:

$$U_c(c, n) = \gamma c^{(1-\sigma)\gamma - 1} (1-n)^{(1-\sigma)(1-\gamma)} \qquad U_n(c, n) = -(1-\gamma) c^{(1-\sigma)\gamma} (1-n)^{(1-\sigma)(1-\gamma)-1}.$$

There is also a utility gain from leaving bequests:

$$v(b) = \chi \frac{(b + b_0)^{\gamma(1-\sigma)}}{1-\sigma} \qquad \text{with } v'(b) = \chi\gamma (b + b_0)^{(1-\sigma)\gamma - 1}$$

Capital demand and profits

The production problem of each agent is static in nature and can be solved in isolation of the other decisions the agent takes. The agent maximizes the profits coming from the entrepreneurial activity by choosing an optimal capital demand:

$$\pi\left(a,z\right)=\max_{k\leq\vartheta a}\mathcal{R}\left(zk\right)^{\mu}-\left(r+\delta\right)k$$

in this expression we are already substituting by the optimal demand for intermediate goods from the final good producer and $z=f\left(z^{p},z^{s}\right)$. The optimal capital demand and corresponding profits are

$$k\left(a,z\right)=\max\left\{ \left(\frac{\mu\mathcal{R}z^{\mu}}{r+\delta}\right)^{\frac{1}{1-\mu}},\vartheta a\right\} \qquad\pi\left(a,z\right)=\begin{cases}\mathcal{R}\left(z\vartheta a\right)^{\mu}-\left(r+\delta\right)\vartheta a & \text{if }k\left(a,z\right)=\vartheta a\\ \left(\frac{1-\mu}{\mu}\right)\left(r+\delta\right)^{\frac{-\mu}{1-\mu}}\left(\mu\mathcal{R}z^{\mu}\right)^{\frac{1}{1-\mu}} & \text{otw}\end{cases}.$$

Non-Labor income

For future reference it will be useful to define the agent's after tax non-labor income:

$$Y\left(a,z\right)=\left(1-\tau_{a}\right)a+\left(ra+\pi\left(a,z\right)\right)\left(1-\tau_{k}\right),$$

along with the marginal income generated by an increase in assets:

$$Y_{a}\left(a,z\right)=\left(1-\tau_{a}\right)+\left(r+\pi_{a}\left(a,z\right)\right)\left(1-\tau_{k}\right),$$

where:

$$\pi_{a}\left(a,z\right)=\begin{cases}\mu\overline{p}\left(z\vartheta\right)^{\mu}a^{\mu-1}-\left(\delta+r\right)\vartheta & \text{if }\vartheta a\leq\left(\frac{\mu\overline{p}z^{\mu}}{r+\delta}\right)^{\frac{1}{1-\mu}}\\ 0 & \text{otw}\end{cases}.$$

Last period of life

The problem of an agent in its last period of life is:

$$\begin{aligned}V_{H}\left(s\right) & = \max_{c,a',n,b}U\left(c,n\right)+v\left(a'\right)\\ \text{s.t.} & \quad \left(1+\tau_{c}\right)c+a'=Y\left(a,z\right)+y^{R}\left(\kappa,e\right)\end{aligned}$$

The solution is to set:

$$c_H(s) = \begin{cases} \frac{Y(a,z)+y^R(\kappa,e)+b_0}{(1+\chi_b)(1+\tau_c)} & \text{if } Y(a,z)+y^R(\kappa,e) > \frac{b_0}{\chi_b} \\ \frac{Y(a,z)+y^R(\kappa,e)}{1+\tau_c} & \text{if } Y(a,z)+y^R(\kappa,e) \leq \frac{b_0}{\chi_b} \end{cases}$$

$$a'_H(s) = \begin{cases} \frac{\chi_b}{1+\chi_b}\left(Y(a,z)+y^R(\kappa,e)\right) - \frac{1}{1+\chi_b}b_0 & \text{if } Y(a,z)+y^R(\kappa,e) > \frac{b_0}{\chi_b} \\ 0 & \text{if } Y(a,z)+y^R(\kappa,e) \leq \frac{b_0}{\chi_b} \end{cases}$$

$$n_H(s) = 0$$

where $\chi_b = \frac{((1+\tau_c)\chi)^{\frac{1}{1-\gamma(1-\sigma)}}}{1+\tau_c}$.

Retirement period

The problem of an agent during retirement is:

$$\begin{aligned} V_h(s) &= \max_{c,a',n} U(c,n) + \beta s_h \sum_{z^s} \Pi_z\left(z^s, z^{s'}\right) V_{h+1}\left(a', z^p, \kappa, e, z^{s'}\right) + (1-s_h)\, v\left(a'\right) \\ \text{s.t.} & \quad (1+\tau_c)\, c + a' = Y(a,z) + y^R(\kappa,e). \end{aligned}$$

The solution for the labor supply decision is to set $n_h(s) = 0$ while the solution for the consumption saving decision is characterized by the Euler equation:

$$\begin{aligned} \frac{1}{1+\tau_c} U_c(c_h, 0) &\geq \beta s_h \sum_{z^s} \Pi_z\left(z^s, z^{s'}\right)\left(\frac{Y_a\left(a', z'\right)}{1+\tau_c} U_c(c_{h+1}, 0)\right) + (1-s_h)\, v'\left(a'\right); \\ a'_h(s) &= Y(a,z) + y^R(\kappa,e) - (1+\tau_c)\, c_h(s). \end{aligned}$$

Given the functional form this results in:

$$\begin{aligned} c_h(s) \leq & \left[\beta s_h \sum_{z^s} \Pi_z\left(z^s, z^{s'}\right)\left(Y_a\left(a', z'\right)\left(c_{h+1}\left(a', z^p, \kappa, e, z^{s'}\right)\right)^{(1-\sigma)\gamma-1}\right)\right. \\ & \left. + (1-s_h)(1+\tau_c)\, v'\left(a'\right)\right]^{\frac{1}{(1-\sigma)\gamma-1}}; \\ a'_h(s) = & \; Y(a,z) + y^R(\kappa,e) - (1+\tau_c)\, c_h(s), \end{aligned}$$

with equality if $a'_h(s) > a_{min}$.

We solve this equation using EGM:

For ages $h \in \left\{H-1, H-2, \ldots, \overline{h}\right\}$:

1. Set an $a' \in \vec{a}$.

2. Define $\tilde{c}_h\left(a^{'}, z^p, \kappa, e, z^s\right)$ as the endogenous consumption implied by $a^{'}$:

$$\tilde{c}_h\left(a^{'}, z^p, \kappa, e, z^s\right) = \left[\beta s_h \sum_{z^s} \Pi_z\left(z^s, z^{s'}\right)\left(Y_a\left(a^{'}, z^{'}\right)\left(c_{h+1}\left(a^{'}, z^p, \kappa, e, z^{s'}\right)\right)^{(1-\sigma)\gamma-1}\right) + (1-s_h)(1+\tau_c)\, v^{'}\left(a^{'}\right)\right]^{\frac{1}{(1-\sigma)\gamma-1}}.$$

Because of backward induction the policy function c_{h+1} is already known.

3. This implies an endogenous value of current non-labor income $\tilde{Y}\left(a^{'}\right)$ given by:

$$\tilde{Y}\left(a^{'}, z^p, \kappa, e, z^s\right) = a^{'} + (1+\tau_c)\,\tilde{c}_h\left(a^{'}, z^p, \kappa, e, z^s\right) - y^R(\kappa, e)$$

   (a) Obtain the the policy function for consumption by using linear interpolation with the values of $\tilde{c}_h$, at known values of $\tilde{Y}$, for the values of $Y(a, z)$ implied by the grid $\vec{a}$.

   (b) The value of $a_h^{'}(s)$ is:

$$a_h^{'}(s) = Y(a, z) + y^R(\kappa, e) - (1+\tau_c)\, c_h(s)$$

   where c_h is obtained through interpolation.

Two things can go wrong in the process above:

1. This process can induce values of $a^{'}$ that lie below the minimum value of the grid a_{min}.

   (a) In that case set $a_h^{'}(s) = a_{min}$ and $c_h(s) = \frac{Y(a,z)+y^R(\kappa,e)-a_{min}}{1+\tau_c}$.

2. The process above can induce extrapolation if there is an $\vec{a}_i$ for which $Y(\vec{a}_i, z) < \tilde{Y}(a_{min}, z)$.

   (a) In this case the Euler equation we solve numerically for $a_h^{'}(\vec{a}_i, z^p, \kappa, e, z^s)$.

   (b) To evaluate the equation, we obtain the value of $c_{h+1}\left(a^{'}, z^p, \kappa, e, z^{s'}\right)$ through interpolation because the values of c_{h+1} are known at the values of the grid $\vec{a}$.

   (c) We use Brent's minimization method (Numerical Recipes, Ch. 10.2) to minimize the following objective function:

$$f\left(a^{'}\right) = \left(\begin{array}{c} Y(\vec{a}_i, z) + y^R(\kappa, e) - a^{'} \\ -\left[\begin{array}{c} \beta s_h \sum_{z^s} \Pi_z\left(z^s, z^{s'}\right)\left(Y_a\left(a^{'}, z\right)\left(c_{h+1}\left(a^{'}, z^p, \kappa, e, z^{s'}\right)\right)^{(1-\sigma)\gamma-1}\right) \\ +(1-s_h)(1+\tau_c)\, v^{'}\left(a^{'}\right)\end{array}\right]^{\frac{1}{(1-\sigma)\gamma-1}}\end{array}\right)^2$$

(d) Once $a_h^{'}(s)$ is obtained consumption is computed as $c_h(s) = \frac{Y(a,z)+y^R(\kappa,e)-a_h^{'}(s)}{1+\tau_c}$.

It is also possible that $a^{'}$ is above a_{max} or $Y(a,z) > \tilde{Y}(a_{max},z)$. This is unlikely to be a problem because the policy functions imply $a^{'} < a$ for large enough a.

Working period

The problem of an agent before retirement is:

$$\begin{aligned} V_h(s) &= \max_{c,a^{'},n} U(c,n) + \beta s_h \sum_{z^{s'},e^{'}} \Pi_z\left(z^s, z^{s'}\right) \Pi_e\left(e, e^{'}\right) V_{h+1}\left(a^{'}, z^p, \kappa, e^{'}, z^{s'}\right) + (1-s_h)\, v\left(a^{'}\right) \\ \text{s.t.} \quad & (1+\tau_c)\, c + a^{'} = Y(a,z) + (1-\tau_l)\, w\hat{y}_h(\kappa,e)\, n \end{aligned}$$

The solution for this problem is characterized by:

$$\begin{aligned} (1-\tau_l)\, w\hat{y}_h(\kappa,e) &\geq \frac{-U_n(c_h,n_h)}{U_c(c_h,n_h)}(1+\tau_c) \\ \frac{U_c(c_h,n)}{1+\tau_c} &\geq \beta s_h \sum_{z^{s'},e^{'}} \Pi_z\left(z^s,z^{s'}\right) \Pi_e\left(e,e^{'}\right) Y_a\left(a^{'},z\right) \frac{U_c(c_{h+1},n_{h+1})}{1+\tau_c} + (1-s_h)\, v^{'}\left(a^{'}\right) \\ a_h^{'}(s) &= Y(a,z) + (1-\tau_l)\, w\hat{y}_h(\kappa,e)\, n_h(s) - (1+\tau_c)\, c_h(s) \end{aligned}$$

Given the functional form this results in:

$$\begin{aligned} c_h(s) &\leq \frac{\gamma(1-\tau_l)\, w\hat{y}_h(\kappa,e)}{(1+\tau_c)(1-\gamma)}(1-n_h) \\ c^{(1-\sigma)\gamma-1}(1-n)^{(1-\sigma)(1-\gamma)} &\geq \beta s_h \sum_{z^{s'},e^{'}} \Pi_z\left(z^s,z^{s'}\right) \Pi_e\left(e,e^{'}\right) \left(Y_a\left(a^{'},z^{'}\right)(c_{h+1})^{(1-\sigma)\gamma-1}(1-n_{h+1})^{(1-\sigma)(1-\gamma)}\right) \\ &\quad + (1-s_h)(1+\tau_c)\, v^{'}\left(a^{'}\right) \\ a_h^{'}(s) &= Y(a,z) + (1-\tau_l)\, w\hat{y}_h(\kappa,e)\, n_h(s) - (1+\tau_c)\, c_h(s) \end{aligned}$$

with equality if $n_h(s) > 0$ and $a_h^{'}(s) > a_{min}$ respectively.

This system is solved using EGM:

For ages $h < \overline{h}$:

1. Set an $a^{'} \in \vec{a}$.

2. Define $\tilde{c}_h\left(a^{'}, z^p, \kappa, e, z^s\right)$ as the endogenous consumption implied by $a^{'}$ if the labor

supply is strictly positive:

$$\tilde{c}_h\left(a^{'}, z^p, \kappa, e, z^s\right) = \left(\frac{\gamma\left(1-\tau_l\right) w\hat{y}_h\left(\kappa, e\right)}{\left(1+\tau_c\right)\left(1-\gamma\right)}\right)^{\frac{-(1-\sigma)(1-\gamma)}{\sigma}}$$

$$\left[\beta s_h \sum_{z^{s'}, e'} \Pi_z\left(z^s, z^{s'}\right) \Pi_e\left(e, e^{'}\right) Y_a\left(a^{'}, z^{'}\right) c_{h+1}^{(1-\sigma)\gamma-1}\left(1-n_{h+1}\right)^{(1-\sigma)(1-\gamma)} + \left(1-s_h\right)\left(1+\tau_c\right) v^{'}\left(a^{'}\right)\right]^{\frac{-1}{\sigma}}.$$

Because of backward induction the policy functions c_{h+1} and n_{h+1} are already known.

3. This consumption implies a labor supply given by:

$$\tilde{n}_h\left(a^{'}, z^p, \kappa, e, z^s\right) = 1 - \frac{\left(1+\tau_c\right)\left(1-\gamma\right)}{\gamma\left(1-\tau_l\right) w\hat{y}_h\left(\kappa, e\right)} \tilde{c}_h\left(a^{'}, z^p, \kappa, e, z^s\right)$$

4. Check if $\tilde{n}_h\left(a^{'}, z^p, \kappa, e, z^s\right) > 0$. If not then:

   (a) Set $\tilde{n}_h\left(a^{'}, z^p, \kappa, e, z^s\right) = 0$.

   (b) Set $\tilde{c}_h\left(a^{'}, z^p, \kappa, e, z^s\right)$ from the Euler equation as:

$$\tilde{c}_h\left(a^{'}, z^p, \kappa, e, z^s\right) \qquad \left[\beta s_h \sum_{z^{s'}, e'} \Pi_z\left(z^s, z^{s'}\right) \Pi_e\left(e, e^{'}\right)\left(Y_a\left(a^{'}, z^{'}\right) c_{h+1}^{(1-\sigma)\gamma-1}\left(1-n_{h+1}\right)^{(1-\sigma)(1-\gamma)}\right) + \left(1-s_h\right)\left(1+\tau_c\right) v^{'}\left(a^{'}\right)\right]^{\frac{1}{(1-\sigma)\gamma-1}}$$

5. This implies an endogenous value of current non-labor income

$$\tilde{Y}\left(a^{'}, z^p, \kappa, e, z^s\right) = a^{'} + \left(1+\tau_c\right) \tilde{c}_h\left(a^{'}, z^p, \kappa, e, z^s\right) - \left(1-\tau_l\right) w\hat{y}_h\left(\kappa, e\right) \tilde{n}_h\left(a^{'}, z^p, \kappa, e, z^s\right)$$

   (a) Obtain the the policy function for consumption by interpolating the values of $\tilde{c}_h$, at known values of $\tilde{Y}$, for the values of $Y\left(a, z\right)$ implied by the grid $\vec{a}$. (We use linear interpolation).

   (b) Obtain the policy function for labor from the first order condition as:

$$n_h\left(s\right) = \max\left\{0\ ,\ 1 - \frac{\left(1+\tau_c\right)\left(1-\gamma\right)}{\gamma\left(1-\tau_l\right) w\hat{y}_h\left(\kappa, e\right)} c_h\left(s\right)\right\}$$

(c) The value of $a_h^{'}(s)$ is:

$$a_h^{'}(s) = Y(a,z) + (1-\tau_l)\, w\hat{y}_h(\kappa,e)\, n_h(s) - (1+\tau_c)\, c_h(s)$$

Two things can go wrong in the process above:

1. This process can induce values of $a^{'}$ that lie below the minimum value of the grid a_{min}.

   (a) Set $a_h^{'}(s) = a_{min}$.

   (b) This implies a labor supply given by:

   $$n_h(s) = \max\left\{0\,,\ \gamma - (1-\gamma)\left(\frac{Y(a,z) - a_{min}}{(1-\tau_l)\, w\hat{y}_h(\kappa,e)}\right)\right\}$$

   (c) This implies a consumption level of:

   $$c_h(s) = \frac{Y(a,z) + (1-\tau_l)\, w\hat{y}_h(\kappa,e)\, n_h(s) - a_h^{'}(s)}{(1+\tau_c)}$$

2. The process above can induce extrapolation if there is an $\vec{a}_i$ for which $Y(\vec{a}_i, z) < \tilde{Y}(a_{min}, z)$. We then solve the Euler equation numerically for $a_h^{'}(\vec{a}_i, z^p, \kappa, e, z^s)$.

   (a) We interpolate to obtain the values of $c_{h+1}\left(a^{'}, z^p, \kappa, e^{'}, z^{s'}\right)$ and $n_{h+1}\left(a^{'}, z^p, \kappa, e^{'}, z^{s'}\right)$ and evaluate the first order condition.

   (b) We use Brent's minimization method (Numerical Recipes, Ch. 10.2) for solving the problem. The objective functions is:

   $$f\left(a^{'}\right) = \left(\begin{array}{c} c^{(1-\sigma)\gamma-1}(1-n)^{(1-\sigma)(1-\gamma)} \\ -\beta s_h \sum_{z^{s'},e'} \Pi_z\left(z^s, z^{s'}\right) \Pi_e\left(e, e^{'}\right) \left(Y_a\left(a^{'}, z^{'}\right)(c_{h+1})^{(1-\sigma)\gamma-1}(1-n_{h+1})^{(1-\sigma)(1-\gamma)}\right) \\ -(1-s_h)(1+\tau_c)\, v^{'}\left(a^{'}\right) \end{array}\right)^2$$

   where:

   $$n = \max\left\{0\,,\ \gamma - (1-\gamma)\left(\frac{Y(a,z) - a^{'}}{(1-\tau_l)\, w\hat{y}_h(\kappa,e)}\right)\right\},$$

   and

   $$c = \frac{Y(a,z) + (1-\tau_l)\, w\hat{y}_h(\kappa,e)\, n - a^{'}}{(1+\tau_c)},$$

   (c) Once we obtain $a_h^{'}(s)$, we compute consumption and labor supply as in the previous step.

As before, $a^{'}$ can also be above a_{max} or $Y(a,z) > \tilde{Y}(a_{max},z)$. This is unlikely to be a problem because the policy functions imply $a^{'} < a$ for large enough a.

1.3 Production

The final good is produced by a public corporate sector and by an aggregator that uses as inputs the output of individual producers. We also consider an extension where the output of the public and private sectors is combined through a Cobb-Douglas technology. See the paper for details.

The Public Corporate Sector

Final good, Y, is produced using a Cobb-Douglas technology with constant returns to scale that combines capital and labor,

$$Y_C = A_C K_C^{\alpha_C} L_C^{1-\alpha_C}$$

The firms in the corporate sector are price takers and their optimal choices are characterized by:

$$r + \delta = \alpha_C A_C \left(\frac{K_C}{L_C}\right)^{-(1-\alpha_C)} \qquad \overline{w} = (1-\alpha_C) A_C \left(\frac{K_C}{L_C}\right)^{\alpha_C}$$

In our baseline we set $A_C = 0$.

The Private Corporate Sector

The final good, Y, is produced according to a Cobb-Douglas technology,

$$Y_P = Q^{\alpha} L_P^{1-\alpha}, \tag{1}$$

where L is effective labor, and Q is the CES composite of intermediate inputs, x_i:

$$Q = \left(\int x_{ih}^{\mu} didh\right)^{1/\mu}. \tag{2}$$

The final goods producing sector is competitive, so the profit maximization problem is

$$\max_{\{\{x_{ih}\},L_P\}} \left(\int x_{ih}^{\mu} didh\right)^{\alpha/\mu} L_P^{1-\alpha} - \int p_{ih} x_{ih} didh - \overline{w} L_P,$$

where p_i is the price of the intermediate good i. The first-order optimality conditions yield the inverse demand (price) function for each intermediate input and the wage rate:

$$p\left(x_{ih}\right)=\alpha x_{ih}^{\mu-1}Q^{\alpha-\mu}L_P^{1-\alpha} \qquad \overline{w}=(1-\alpha)Q^{\alpha}L_P^{-\alpha}. \tag{3}$$

Intermediate Goods Producers

There is a continuum of intermediate goods, each produced by a different individual according to a linear technology,

$$x_{ih}=z_{ih}k_{ih}, \tag{4}$$

where k_{ih} is the final good (consumption/capital) used in production by entrepreneur i, and z_{ih} is her stochastic and idiosyncratic entrepreneurial productivity at age h. For clarity, we suppress the subscripts i and h when possible.

Every period, the individual/entrepreneur chooses the optimal capital level to maximize profit:

$$\pi\left(a,z\right)=\max_{k\leq\vartheta(z)a}\left\{p\left(zk\right)\times zk-\left(r+\delta\right)k\right\}, \tag{5}$$

where δ is the depreciation rate. The price of the differentiated good in (3) can be written as: $p\left(zk\right)=\mathcal{R}\times\left(zk\right)^{\mu-1}$, where $\mathcal{R}\equiv\alpha Q^{\alpha-\mu}L_P^{1-\alpha}$, yielding the solution

$$k\left(a,z\right)=\min\left\{\left(\frac{\mu\mathcal{R}z^{\mu}}{r+\delta}\right)^{\frac{1}{1-\mu}},\vartheta(z)a\right\}, \tag{6}$$

with the associated maximized profit function

$$\pi\left(a,z\right)\equiv\begin{cases}\mathcal{R}\left(z\vartheta(z)a\right)^{\mu}-\left(r+\delta\right)\vartheta(z)a & \text{if } k\left(a,z\right)=\vartheta(z)a\\ \left(1-\mu\right)\mathcal{R}z^{\mu}\left(\frac{\mu\mathcal{R}z^{\mu}}{r+\delta}\right)^{\frac{\mu}{1-\mu}} & \text{if } k\left(a,z\right)<\vartheta(z)a\end{cases}. \tag{7}$$

1.4 Prices

Given a distribution Γ and policy functions for consumption, labor, saving and capital demand one can update the aggregate prices of the economy.

1. Guess a level of Q and a level of wage $\overline{w}$.
2. From the optimality condition of the private corporate sector obtain the demand for labor: $L_P=\left(\frac{1-\alpha}{\overline{w}}\right)^{\frac{1}{\alpha}}Q$.
3. Using Q and L_P obtain $\mathcal{R}=\alpha Q^{\alpha-\mu}L_P^{1-\alpha}$.

4. From the optimality conditions of the public corporate sector obtain that sector's ratio of capital to labor: $\frac{K_C}{L_C} = \left(\frac{\overline{w}}{(1-\alpha_C)A_C}\right)^{\frac{1}{\alpha_C}}$.

5. From the optimality conditions of the public corporate sector obtain the interest rate: $r = \alpha_C A_C \left(\frac{K_C}{L_C}\right)^{-(1-\alpha_C)} - \delta$.

6. Given r and $\mathcal{R}$ solve the problem of the intermediate goods producers. Use the distribution Γ to aggregate the demand for capital in the private corporate sector:

$$K_P = \sum_{h=1}^{H} \sum_{a,z^p,\kappa,e,z^s} k\left(a,z\right)\Gamma\left(h,a,z^p,\kappa,e,z^s\right)$$

7. Given the solution to the intermediate goods producers problem and $\overline{w}$, solve the dynamic programming problem of the consumers. Use the distribution Γ to aggregate the supply of assets (capital) and labor:

$$A = \sum_{h=1}^{H} \sum_{a,z^p,\kappa,e,z^s} a\Gamma\left(h,a,z^p,\kappa,e,z^s\right)$$
$$L^S = \sum_{h=1}^{\overline{h}-1} \sum_{a,z^p,\kappa,e,z^s} \left(\hat{y}_h\left(\kappa,e\right) n_h\left(a,z^p,\kappa,e,z^s\right)\right)\Gamma\left(h,a,z^p,\kappa,e,z^s\right)$$

   (a) Given the solution to the dynamic programming problem the distribution can be updated by forward iteration following Young (2010)'s histogram method.

8. Capital Market Clearing

   (a) If $K_P < A$ then $K_C = A - K_P$.

   (b) if $K_P > A$ then see below.

9. Labor demand

   (a) If the capital market cleared then obtain labor demand in the public corporate sector as: $L_C = \left(\frac{\overline{w}}{(1-\alpha_C)A_C}\right)^{\frac{-1}{\alpha_C}} K_C$. Then Total labor demand is:

$$L^D = L_P + L_C = \left(\frac{1-\alpha}{\overline{w}}\right)^{\frac{1}{\alpha}} Q + \left(\frac{(1-\alpha_C)A_C}{\overline{w}}\right)^{\frac{1}{\alpha_C}} K_C$$

   (b) If the capital market did not clear then see below.

10. **Update Prices and Quantities**

    (a) If capital market cleared.

    i. Given the solution to the intermediate goods producers problem and the distribution Γ obtain the new value of Q:

$$Q^{'} = \left(\sum_{h=1}^{H} \sum_{a,z^p,\kappa,e,z^s} (zk(a,z))^{\mu} \Gamma(h,a,z^p,\kappa,e,z^s) \right)^{\frac{1}{\mu}}$$

    ii. Given the current distribution Γ, the current labor supply L^S, the current public corporate sector capital and the new value for $Q^{'}$ choose $\overline{w}^{'}$ to clear the labor market:

$$L^S = \left(\frac{1-\alpha}{\overline{w}^{'}} \right)^{\frac{1}{\alpha}} Q^{'} + \left(\frac{(1-\alpha_C) A_C}{\overline{w}^{'}} \right)^{\frac{1}{\alpha_C}} K_C$$

    If $\alpha_C = \alpha$ then:

$$\overline{w}^{'} = \left[\frac{(1-\alpha)^{\frac{1}{\alpha}} Q^{'} + ((1-\alpha_C) A_C)^{\frac{1}{\alpha_C}} K_C}{L^S} \right]^{\alpha}$$

    (b) If capital market did not clear the initial Q was too high. Set $Q^{'} = Q/2$ and leave $\overline{w}^{'} = \overline{w}$.

Without a public corporate sector $(A_C = 0)$ the prices are updated differently

1. The wage is obtained by directly evaluating the first order condition of the final good firm:

$$w = (1-\alpha) \left(\frac{Q}{N} \right)^{\alpha}$$

   where:

$$Q = \left(\sum_{h=1}^{H} \sum_{a,z^p,\kappa,e,z^s} (zk(a,z))^{\mu} \Gamma(h,a,z^p,\kappa,e,z^s) \right)^{\frac{1}{\mu}}$$

$$N = \sum_{h=1}^{\overline{h}-1} \sum_{a,z^p,\kappa,e,z^s} (\hat{y}_h(\kappa,e)\, n_h(a,z^p,\kappa,e,z^s))\, \Gamma(h,a,z^p,\kappa,e,z^s)$$

2. The average labor income (needed for computing retirement income) is given by:

$$\overline{E} = \frac{wN}{\sum_{h=1}^{\overline{h}-1} \sum_{a,z^p,\kappa,e,z^s} \Gamma\left(h, a, z^p, \kappa, e, z^s\right)}$$

3. The parameter $\mathcal{R}$ in the agent's profit maximization problem is given by:

$$\mathcal{R} = \alpha Q^{\alpha-\mu} N^{1-\alpha}$$

   (a) In the process of converging to the stationary distribution $\mathcal{R}$ can explode to infinity so in practice the formula above is capped by 1. This constraint does not bind as we achieve convergence.

4. The interest rate has no closed form solution and has to be found numerically.

   (a) The interest rate must be such that the demand and supply of capital are equalized.

   (b) The supply of capital is given by the total assets in the economy:

$$A = \sum_{h=1}^{H} \sum_{a,z^p,\kappa,e,z^s} a\Gamma\left(h, a, z^p, \kappa, e, z^s\right)$$

   (c) The demand for capital is given by:

$$K\left(r\right) = \sum_{h=1}^{H} \sum_{a,z^p,\kappa,e,z^s} k\left(a, z|r\right) \Gamma\left(h, a, z^p, \kappa, e, z^s\right)$$

   where the individual capital demand depends also on the value of the interest rate.

   (d) We use Brent's minimization method (Numerical Recipes, Ch. 10.2) for solving the problem. The objective functions is:

$$f\left(r\right) = \left(\frac{K\left(r\right)}{A} - 1\right)^2$$

We hold all policy functions and other aggregates constant when finding the prices. In particular, when finding the interest rate that clears the market the value of $\mathcal{R}$ is not changed, although the capital demand also depends on $\mathcal{R}$.

1.5 Distribution

The algorithm to find the stationary distribution is as follows:

1. Guess an initial distribution and initial prices:

   In practice we use set the initial distribution to have full support on a_{min} with uniform weights and assume that both z and κ are drawn from their stationary distribution, e behaves like in G_e^h, and population by age is like in Bell and Miller (2002):

$$\Gamma^0(h, a_{min}, z, \kappa, e) = \frac{\text{pop}_h}{\sum \text{pop}_h} G_z(z) G_\kappa(\kappa) G_e^h(h, e)$$

2. Solve for policy functions given current prices.

3. Use Young (2010) histogram method to iterate forward the distribution to get Γ^{i+1}. Repeat this n times.

   In practice we use $n = 5$.

   (a) Discretization of policy function: It is necessary to map the continuous policy function of assets to the discrete grid $\vec{a}$ on which the distribution is defined.

      i. For each age h and state $s = (a, z^p, \kappa, e, z^s)$ define:

$$a_h^{lo}(s) = \max\left\{a_i \in \vec{a} | a_i \leq a_h^{'}(s)\right\}$$

      that is the element of the grid to the left of $a_h^{'}(s)$.

      ii. If $a_h^{'}(s) < a_{min}$ then set $a_h^{lo}(s) = a_{min}$.

      iii. If $a_h^{'}(s) > a_{max}$ then set $a_h^{lo}(s) = \vec{a}_{na-1}$.

      iv. Define $a_h^{hi}(s)$ as the element of the grid to the right of $a_h^{'}(s)$ (obtained by summing one to the index of $a_h^{lo}(s)$).

      v. Define the weight (probability) on $a_h^{lo}(s)$ as the relative distance between the optimal action and the grid node.:

$$\Pr\left(a_h^{lo}(s)\right) = \frac{a_h^{hi}(s) - a_h^{'}(s)}{a_h^{hi}(s) - a_h^{lo}(s)}$$

      This probability must be adjusted to be between 0 and 1 if necessary.

      vi. Define the weight on $a_h^{hi}(s)$ as $\Pr\left(a_h^{hi}(s)\right) = 1 - \Pr\left(a_h^{lo}(s)\right)$.

   (b) Update the distribution.

i. Agents of age H die and are replaced by agents of age 1 with no assets, $e = 1$, $z^s = H$, and z and κ that evolve according to P_z and P_κ. $a = a_{min}$.

ii. Agents with age $h \geq \overline{h}$ have two options:

A. If they don't die, they become agents of age $h + 1$ and they don't change their states (z^p, κ, e). Their a is updated according to the discretization described above. z^s evolves according to the transition matrix Π_z.

B. If they die they, they become agents of age 1 and $e = 1$, $z^s = H$, z and κ that evolve according to P_z and P_κ. Their a is updated according to the discretization described above. (saving occurs before death).

iii. Agents in working age are updated in the same way, but their e state has to be updated if they continue living.

4. Use the new distribution to update prices.

5. Repeat steps 2–4 until convergence in the distribution is achieved.

$$\max\left(\left|\Gamma^{i+1}\left(h, a, z^p, \kappa, e, z^s\right) - \Gamma^{i}\left(h, a, z^p, \kappa, e, z^s\right)\right|\right) < 10^{-8}$$

1.6 Value function

The value function can be computed in a straight-forward manner from the policy functions of the agents. The value function is solved by backward induction, and we use linear interpolation to compute the future value function at the optimal $a^{'}$.

1.7 Tax reform

1. Solve the model setting $\tau_k = 0.25$ and $\tau_a = 0$.

2. Compute the value function and store it.

3. Compute government revenue net of social security transfers and store it:

$$
\begin{aligned}
G + SSP = {} & \tau_k \int_{h,a,\mathbf{S}} (ra + \pi(a,z)) \, d\Gamma(h,a,\mathbf{S}) \\
& + \tau_a \int_{h,a,\mathbf{S}} (a) \, d\Gamma(h,a,\mathbf{S}) \\
& + \tau_\ell \int_{h<R,a,\mathbf{S}} (\overline{w} w_h(\kappa,e) \, \ell_h(a,\mathbf{S})) \, d\Gamma(h,a,\mathbf{S}) \\
& + \tau_c \int_{h,a,\mathbf{S}} c_h(a,\mathbf{S}) \, d\Gamma(h,a,\mathbf{S}), \\
& + \tau_b \int_{h,a,\mathbf{S}} (1 - s_{h+1}) \, a_{h+1}(a,\mathbf{S}) \, d\Gamma(h,a,\mathbf{S}),
\end{aligned} \tag{8}
$$

4. Set $\tau_k = 0$ and $\tau_a = 0.01$, solve the model and compute government revenue again.
5. While $\overline{G}_w < \overline{G}$ set $\tau_a = 0.005 + \tau_a$ solve the model and compute government revenue.
   (a) This is bracketing the level of wealth taxes for which $\overline{G}_w = \overline{G}$.
   (b) Let $\overline{\tau}_w$ be the first wealth for which $\overline{G}_w > \overline{G}$, and $\underline{\tau}_w = \overline{\tau}_w - 0.005$.
   (c) Make a guess for the wealth tax for which $\overline{G}_w = \overline{G}$ as:

$$\tau_a = \underline{\tau}_w + \frac{\overline{G} - \overline{G}_w(\underline{\tau}_w)}{\overline{G}_w(\overline{\tau}_w) - \overline{G}_w(\underline{\tau}_w)} \overline{\tau}_w$$

6. Use bisection to find the wealth tax τ_a such that $\overline{G}_w = \overline{G}$.
   (a) If $\overline{G}_w > \overline{G}$ then: $\overline{\tau}_w = \tau_a$.
   (b) If $\overline{G}_w \leq \overline{G}$ then $\underline{\tau}_w = \tau_a$.
   (c) Let $\tau_a = \frac{\underline{\tau}_w + \overline{\tau}_w}{2}$, solve the model and compute $\overline{G}_w$
   (d) Repeat while $\left| \frac{\overline{G}_w}{\overline{G}} - 1 \right| > 0.00001$.
7. Compute the value function and perform welfare analysis.

1.8 Optimal taxes

Optimal taxes are found by maximizing the average value of a newborn. I first describe how the objective function is evaluated in the case of capital income taxes. Let $f(\tau_k)$ be the objective function.

1. Solve the model setting $\tau_k = 0.25$ and $\tau_a = 0$.

2. Compute government revenue net of social security transfers and store it.

3. Fix a value for τ_k to evaluate the objective function at.

4. Guess a value for τ_l.

5. Solve for the model and compute $G(\tau_k)$ as in step 2.

6. While $\left|\frac{G(\tau_k)}{\overline{G}} - 1\right| > 0.00001$.

   (a) Update τ_l as:

   $$\tau_l^{'} = \tau_l - \frac{G(\tau_k) - \overline{G}}{\sum_{h=1}^{H} \sum_{s=(a,z^p,\kappa,e,z^s)} [w\hat{y}_h(\kappa, e) n_h(s)] \Gamma(h, s)}$$

   This corresponds to the value of τ_l that would guarantee $G(\tau_k) = \overline{G}$ given the current policy functions.

      i. In practice we dampen the adjustment and set:

      $$\tau_l = \frac{\tau_l + \tau_l^{'}}{2}$$

   (b) Solve for the model and compute $G(\tau_k)$ as in step 2.

7. Evaluate the value function and compute the objective function as:

   $$f(\tau_k) = \sum_{s=(a,z^p,\kappa,e,z^s)} V_1(s) \frac{\Gamma(1, s)}{\sum_s \Gamma(1, s)}$$

2 Simulation

The simulation is not needed to solve the model as well as most statistics we report that are computed directly from the stationary distribution of agents as described in Ocampo & Robinson (2022). However, we implement a simulation to compute additional statistics from the solution. We simulate and track a cohort of $N = 20,000,000$ dynasties for $T = 2000$ periods. Because of memory requirements the results are not stored as a panel, but as a cross-section. Only the final simulation period is used to compute statistics. Some statistics are supplemented by using repeated cross-sections 50 years apart at the end of the sample.

1. Initialize the cohort. For each agent in the cohort $(i = 1, \ldots, N)$:

(a) Draw an age (h), and exogenous states (z^p, κ, e) from their stationary distributions.

(b) Set $z^p = H$ and $a = a_{min}$ (for all agents).

2. For $t = 1, \ldots, T$ update the states as follows:

   (a) Set $a = a_h^{'}(s)$, where the savings policy function is evaluated at the current states using linear interpolation (a need not be on the grid).

   (b) Draw survival shock from survival probabilities.

      i. If the agent survives set $h = h + 1$ and update z^s using Π_z. If $h + 1 < \overline{h}$ then also update e using P_e.

      ii. If the agent does not survive set $h = 1$ and update z and κ using P_z and P_κ. Set $z^s = H$ and $e = 1$.

3. Policy and value functions can be evaluated for each agent in the cohort once the simulation ends.

3 Progressive labor income taxes

Consider now labor taxes such that the after tax labor income is given by:$(1 - \tau_l)\left(w\hat{y}_h(\kappa, e)\, n_h(s)\right)^{\psi}$. The following parts of the above procedures need to be modified.

Policy Functions - Working period

The first order condition for labor is now:

$$\psi(1 - \tau_l)\left(w\hat{y}_h(\kappa, e)\right)^{\psi} n_h^{\psi-1} \frac{U_c(c_h, n_h)}{1 + \tau_c} \geq -U_n(c_h, n_h)$$

Given the functional form this results in:

$$c_h(s) \leq \frac{\gamma\psi(1 - \tau_l)\left(w\hat{y}_h(\kappa, e)\right)^{\psi} n^{\psi-1}}{(1 + \tau_c)(1 - \gamma)}(1 - n_h)$$

with equality if $n_h(s) > 0$. This condition can no longer be solved directly for labor as a function of consumption so a different approach is taken.

This system is solved using EGM:

For ages $h < \overline{h}$:

1. Set an $a^{'} \in \vec{a}$.

2. Let $\underline{n} > 0$ be a lower bound on the interior solution for hours and define two alternative consumption levels:

$$\tilde{c}_1 = \left[\beta s_h \sum_{z^{s'}, e'} \Pi_z \left(z^s, z^{s'} \right) \Pi_e \left(e, e' \right) \left(Y_a \left(a', z' \right) (c_{h+1})^{(1-\sigma)\gamma - 1} (1 - n_{h+1})^{(1-\sigma)(1-\gamma)} \right) \right.$$
$$\left. + \beta (1 - s_h) v' \left(a' \right) \right]^{\frac{1}{(1-\sigma)\gamma - 1}}$$
$$\tilde{c}_2 = \frac{\gamma \psi (1 - \tau_l) (w \hat{y}_h (\kappa, e))^{\psi} \underline{n}^{\psi - 1}}{(1 + \tau_c)(1 - \gamma)} (1 - \underline{n})$$

note that the first consumption level follows from the Euler equation if $n = 0$.

3. If $\tilde{c}_1 \geq \tilde{c}_2$ then set $\tilde{c}_h \left(a', z^p, \kappa, e, z^s \right) = \tilde{c}_1$ and and $\tilde{n}_h \left(a', z^p, \kappa, e, z^s \right) = 0$.

4. If $\tilde{c}_1 < \tilde{c}_2$ then there is an interior solution for the labor supply $\tilde{n}_h \left(a', z^p, \kappa, e, z^s \right)$. The solution must be obtained numerically. This is done using Brent's method (Numerical Recipes, Ch. 10.2). The objective function is the Euler equation where the consumption is given by the labor supply condition:

$$f(n) = \left(\begin{array}{c} \left(\frac{\gamma \psi (1-\tau_l)(w \hat{y}_h(\kappa, e))^{\psi} n^{\psi - 1}}{(1+\tau_c)(1-\gamma)} \right)^{(1-\sigma)\gamma - 1} (1 - n)^{-\sigma} \\ - \left[\begin{array}{c} \beta s_h \sum_{z^{s'}, e'} \Pi_z \left(z^s, z^{s'} \right) \Pi_e \left(e, e' \right) Y_a \left(a', z' \right) c_{h+1}^{(1-\sigma)\gamma - 1} (1 - n_{h+1})^{(1-\sigma)(1-\gamma)} \\ + \beta (1 - s_h) v' \left(a' \right) \end{array} \right] \end{array} \right)^2$$

5. Define $\tilde{c}_h \left(a', z^p, \kappa, e, z^s \right)$ as the endogenous consumption implied by a' and $\tilde{n}_h \left(a', z^p, \kappa, e, z^s \right)$:

$$\tilde{c}_h \left(a', z^p, \kappa, e, z^s \right) = \frac{\gamma \psi (1 - \tau_l) (w \hat{y}_h (\kappa, e))^{\psi} \tilde{n}_h \left(a', z^p, \kappa, e, z^s \right)^{\psi - 1}}{(1 + \tau_c)(1 - \gamma)} \left(1 - \tilde{n}_h \left(a', z^p, \kappa, e, z^s \right) \right)$$

6. This implies an endogenous value of current non-labor income $\tilde{Y} \left(a' \right)$ given by:

$$\tilde{Y} \left(a', z^p, \kappa, e, z^s \right) = a' + (1 + \tau_c) \tilde{c}_h \left(a', z^p, \kappa, e, z^s \right) - (1 - \tau_l) \left(w \hat{y}_h (\kappa, e) \tilde{n}_h \left(a', z^p, \kappa, e, z^s \right) \right)^{\psi}$$

   (a) Obtain the the policy function for consumption by interpolating the values of $\tilde{c}_h$, at known values of $\tilde{Y}$, for the values of $Y(a, z)$ implied by the grid $\vec{a}$.

      i. We use linear interpolation.

   (b) Obtain the policy function for labor from the first order condition numerically by minimizing the following function:

$$f(n) = \left(c_h(s) - \frac{\gamma\psi(1-\tau_l)(w\hat{y}_h(\kappa,e))^{\psi} n^{\psi-1}}{(1+\tau_c)(1-\gamma)}(1-n) \right)^2$$

(c) The value of $a_h^{'}(s)$ is obtained as:

$$a_h^{'}(s) = Y(a,z) + (1-\tau_l)(w\hat{y}_h(\kappa,e)n_h(s))^{\psi} - (1+\tau_c)c_h(s)$$

Two things can go wrong in the process above:

1. This process can induce values of $a^{'}$ that lie below the minimum value of the grid a_{min}.

   (a) Set $a_h^{'}(s) = a_{min}$.

   (b) Labor supply is found by numerically solving the first order condition. For this we use Brent (as in all cases) on the following objective function:

$$f(n) = \left(\begin{array}{c} \left(Y(a,z) + (1-\tau_l)(w\hat{y}_h(\kappa,e)n_h(s))^{\psi} - a_{min}\right) \\ -\frac{\gamma\psi(1-\tau_l)(w\hat{y}_h(\kappa,e))^{\psi}n^{\psi-1}}{(1+\tau_c)(1-\gamma)}(1-n) \end{array} \right)^2$$

   (c) This implies a consumption level of:

$$c_h(s) = \frac{Y(a,z) + (1-\tau_l)(w\hat{y}_h(\kappa,e)n_h(s))^{\psi} - a_h^{'}(s)}{(1+\tau_c)}$$

2. The process above can induce extrapolation if there is an $\vec{a}_i$ for which $Y(\vec{a}_i, z) < \tilde{Y}(a_{min}, z)$.

   (a) In this case the Euler equation is solved numerically for $a_h^{'}(\vec{a}_i, z^p, \kappa, e, z^s)$.

   (b) To evaluate the equation the value of $c_{h+1}\left(a^{'}, z^p, \kappa, e, z^{s'}\right)$ and $n_{h+1}\left(a^{'}, z^p, \kappa, e, z^{s'}\right)$ are obtained through interpolation.

   (c) Brent's minimization method (Numerical Recipes, Ch. 10.2) is used for solving the problem the objective functions is:

$$f\left(a^{'}\right) = \left(\begin{array}{c} c^{(1-\sigma)\gamma-1}(1-n)^{(1-\sigma)(1-\gamma)} \\ -\beta s_h \sum_{z^{s'},e'} \Pi_z\left(z^s, z^{s'}\right)\Pi_e\left(e, e^{'}\right)Y_a\left(a^{'}, z^{'}\right)c_{h+1}^{(1-\sigma)\gamma-1}(1-n_{h+1})^{(1-\sigma)(1-\gamma)} \\ -\beta(1-s_h)v^{'}\left(a^{'}\right) \end{array} \right)^2$$

where n is obtained numerically as in the previous case and:

$$c = \frac{Y(a,z) + (1-\tau_l)(w\hat{y}_h(\kappa,e)\,n)^{\psi} - a'}{(1+\tau_c)}$$

(d) Once $a_h'(s)$ is obtained consumption and labor supply are computed as in the previous step.

Government revenue

The government labor tax revenue is computed as:

$$\sum_{h=1}^{H} \sum_{s=(a,z^p,\kappa,e,z^s)} \left[w\hat{y}_h(\kappa,e)\,n_h(s) - (1-\tau_l)(w\hat{y}_h(\kappa,e)\,n_h(s))^{\psi} \right] \Gamma(h,s)$$

Optimal taxes

The objective function now has two arguments.

1. Solve the model setting $\tau_k = 0.25$ and $\tau_a = 0$.
2. Compute government revenue net of social security transfers and store it.
3. Fix a value for τ_k and ψ to evaluate the objective function at.
4. Guess a value for τ_l.
5. Solve for the model and compute $G(\tau_k)$ as in step 2.
6. While $\left| \frac{G(\tau_k)}{\overline{G}} - 1 \right| > 0.00001$.

   (a) Update τ_l as:

   $$\tau_l' = \tau_l - \frac{G(\tau_k) - \overline{G}}{\sum_{h=1}^{H} \sum_{s=(a,z^p,\kappa,e,z^s)} [w\hat{y}_h(\kappa,e)\,n_h(s)]^{\psi}\, \Gamma(h,s)}$$

   This corresponds to the value of τ_l that would guarantee $G(\tau_k) = \overline{G}$ given the current policy functions.

   i. In practice we dampen the adjustment and set:

   $$\tau_l = \frac{\tau_l + \tau_l'}{2}$$

(b) Solve for the model and compute $G(\tau_k)$ as in step 2.

7. Evaluate the value function and compute the objective function as:

$$f(\tau_k, \psi) = \sum_{s=(a,z^p,\kappa,e,z^s)} V_1(s) \frac{\Gamma(1,s)}{\sum_s \Gamma(1,s)}$$

4 Wealth tax threshold

Consider now a case where wealth taxes are progressive and take the form of a piecewise linear function on assets The after tax non-labor income is defined as:

$$Y(a,z) = \begin{cases} a + (ra + \pi(a,z))(1-\tau_k) & \text{if } a \leq \overline{a} \\ a - \tau_a(a-\overline{a}) + (ra + \pi(a,z))(1-\tau_k) & \text{if } a > \overline{a} \end{cases}$$

where $\overline{a}$ is the threshold level for the tax, and τ_a is the tax rate above the threshold.

This tax schedule introduces a non-convexity into the agent's problem, since the marginal benefit of an extra unit of assets is not differentiable for $a = \overline{a}$. Otherwise the problem is just as before. In general the threshold can depend on total before tax non-labor income $Y(a,z)$. We write the problem below in that way. To deal with this non-convexity we modify the algorithm for finding the policy functions as follows:

1. For each z find a_z such that $Y(a,z) = \overline{Y}$.

2. Let $\vec{a}_z = \vec{a} \cup \{a_z\}_{z=1}^{n_z}$ be the expanded grid that contains the original grid nodes and the grid nodes for which $Y(a,z) = \overline{Y}$.

3. Set $a' \in \vec{a}_z$.

4. If $Y(a', z') \neq \overline{Y}$ for all z' reached with positive probability then apply the EGM as described above.

5. If $Y(a', z') = \overline{Y}$ for any z' reached with positive probability then apply the EGM twice.

   (a) For z' such that $Y(a', z') = \overline{Y}$ we solve for the policy functions given no wealth tax .

   (b) For z' such that $Y(a', z') = \overline{Y}$ we solve the for the policy functions given a tax τ_a.

(c) These policy functions describe an inaction region for which the agent chooses to stay just at the threshold.

6. The rest of the EGM procedure follows as before by interpolating to get policy functions on the exogenous grid.

5 Transition

We now consider the problem of a transitioning economy that starts in steady state and reacts to a tax reform. We first describe the transition when moving from the benchmark tax policy (τ_k, τ_a, τ_l), to a new one with taxes τ_k^{∞}, τ_a^{∞}, τ_ℓ^{∞}. The change is unexpected and permanent. The government can borrow and save along the transition to finance any budget surplus or deficit. After describing this transition we discuss how to choose the value of new taxes to ensure that the government budget is balanced in the limit.

To make the explanation of the transition clearer we slightly change the notation used above. Let $V_t^i(\mathbf{s})$ be the value function of an agent in state $\mathbf{s} = (h, a, z^p, \kappa, e, z^s)$ in period t of the transition during iteration i of the algorithm. Note that the state vector $\mathbf{s}$ includes the age of the agent. We let $t = 0$ be the initial steady state and $t = T$ the final period of the transition (for which all variables have reached the steady state under the new tax policy). The same goes for policy functions $\left(\text{e.g. } a_t^{\prime i}(\mathbf{s})\right)$ and the distribution of agents $(\Gamma_t^i(\mathbf{s}))$.

5.1 Initialization of algorithm

To initialize the algorithm we first solve it assuming that the government can borrow from third party sources that lend at the same interest rate as the spot bond market in the economy.

1. Compute the steady state solution of the model under the initial tax policy (τ_k, τ_a, τ_l) and the final tax policy $(\tau_k^{\infty}, \tau_a^{\infty}, \tau_\ell^{\infty})$

2. Guess a path for the aggregate variables $\{Q_t^0, L_t^0, r_t^0\}_{t=0}^{T}$ that are consistent with the initial and final steady states.

   (a) The first time we perform this step we use a linear transition path between steady state values for each variable.

   (b) Otherwise, use the path from the previous iteration.

3. Define paths for $\left\{Y_t^i, \mathcal{R}_t^i, \bar{w}_t^i, \overline{y}_t^i\right\}_{t=0}^T$ consistent with the equilibrium:

$$Y_t^i = \left(Q_t^i\right)^{\alpha}\left(L_t^i\right)^{1-\alpha} \qquad \mathcal{R}_t^i = \alpha\left(Q_t^i\right)^{\alpha-\mu}\left(L_t^i\right)^{1-\alpha} \qquad \overline{w}_t^i = (1-\alpha)\left(Q_t^i\right)^{\alpha}\left(L_t^i\right)^{-\alpha} \qquad \overline{y}_t^i = \frac{\overline{w}_t^i L_t^i}{I_R}$$

where I_R is the fraction of the population below retirement age (R).

4. Solve the dynamic programming problem of the household for all states ($\mathbf{s}$) and periods of the transition ($t = 1, \ldots, T$). The problem is solved by backwards induction and provides a sequence of value and policy functions $\left\{V_t^i(\mathbf{s}), a_t^{\prime i}(\mathbf{s}), \ldots\right\}_{t=1}^T$.

   The problem takes as given the path for $\left\{Q_t^{i-1}, L_t^{i-1}, r_t^{i-1}, Y_t^{i-1}, \mathcal{R}_t^{i-1}, \bar{w}_t^{i-1}, \overline{y}_t^{i-1}\right\}_{t=0}^T$.

   This is the lengthiest step in the procedure.

5. Solve forward the transition path of the distribution $\{\Gamma_t^i(\mathbf{s})\}_{t=1}^T$ given the policy function for savings $\left\{a_t^{\prime i}(\mathbf{s})\right\}_{t=0}^T$ taking as initial condition the initial steady state distribution.

6. Compute government debt at the end of the periods. Debt is initialized at zero (Db $= 0$) and is updated recursively for $t = 1, \ldots, T$ as:

$$\mathrm{Db}_t = (1 + r_t)\,\mathrm{Db}_{t-1} + \left(\underbrace{G_0 - G_t}_{\text{Deficit}}\right),$$

   where G_0 is the government's revenue (and expenditure) in the initial steady state, and G_t is the government revenue in period t. The interest rate on debt is taken from the current period. Government operates at the end of the period and this debt is taken out of the savings of the agents in the economy to pay the deficit and roll-over old debt.

7. Compute path for aggregate variables $\{Q_t^i, L_t^i\}_{t=1}^T$ implied by policy functions and distribution.

$$Q_t^i = \left(\int \left(x_t^i(\mathbf{s})\right)^{\mu} d\Gamma_t^i(\mathbf{s})\right)^{1/\mu} \qquad L_t^i = \int w(\mathbf{s})\,\ell_t^i(\mathbf{s})\,d\Gamma_t^i(\mathbf{s})$$

8. Re-do Step (3) to update $\left\{Y_t^i, \mathcal{R}_t^i, \bar{w}_t^i, \overline{y}_t^i\right\}_{t=0}^T$.

9. Compute path for interest rate $\{r_t^i\}_{t=0}^T$ that clears the intra-period bond market. For

every t choose r_t^i that solves:

$$\int \left(a - k^d \left(a, z^p, z^s; r_t^i, \mathcal{R}_t^i\right)\right) d\Gamma_t^i \left(\mathbf{s}\right) = 0,$$

where $k^d \left(a, z^p, z^s; r_t^i, \mathcal{R}_t^i\right)$ is the optimal demand of capital of a firm with assets a and productivity $\left(z^P, z^s\right)$ at the (already updated prices) $\{\mathcal{R}_t^i\}$ and the new interest rate r_t^i.

(a) The supply of assets is kept constant with respect to r_t^i, and that the problem is solved statically, without taking into account the effects of future and past interest rates on asset accumulation. The demand of assets depends on the updated prices $(w, \mathcal{R})$ for the current period.

10. Repeat (4)-(9) until convergence in terms of the change in the distribution or the aggregate variables Q and L.

11. Compute the government's steady state deficit:

$$G_T - G_0 - r\text{Db}$$

Adjust taxes (say labor income taxes) to close the steady state deficit. Repeat algorithm under new taxes.

The output of this algorithm provides a starting point for the next step. The solution is approximate because it is ignoring the effect of the government in the capital market.

5.2 Transition algorithm with closed financial markets

The algorithm works just as before but it replaces step (9) with:

9′. Compute path for interest rate $\{r_t^i\}_{t=0}^T$ that clears the intra-period bond market taking into account the government's demand for assets.

$$\int \left(a - k^d \left(a, z^p, z^s; r_t^i, \mathcal{R}_t^i\right)\right) d\tilde{\Gamma}_t^i \left(\mathbf{s}; r_t^i\right) - \omega \text{Db}_{t-1} = 0,$$

where ω is a dampening factor that we progressively increase from 0 to 1.

(a) The demand for capital does not depend on the inter-temporal choices of the agents. It is completely determined by the current period prices $(\mathcal{R}_t^i, r_t^i)$.

(b) The choice of r_t^i affects the consumption-saving problem of agents and through it the current supply of assets. To capture this we solve the following auxiliary problem for each state $\mathbf{s} = (h, a, z^p, \kappa, e, z^s)$

$$\max_{\tilde{c}_{t-1}, \tilde{n}_{t-1}, \tilde{a}'_{t-1}} \quad U(\tilde{c}_{t-1}, \tilde{n}_{t-1}) + \beta s_h E_{e', z^{s'}} \left[\tilde{V}\left(h+1, \tilde{a}'_{t-1}, z^p, \kappa, e', z^{s'}; \mathcal{R}_t^i, r_t^i, c_t^i, n_t^i\right)\right]$$
$$\text{s.t.} \quad (1+\tau_c)\,\tilde{c}_{t-1} + \tilde{a}'_{t-1} = \tilde{Y}\left(a, z; \mathcal{R}_{t-1}^i, r_{t-1}^i\right) + Y^n\left(\tilde{n}_{t-1}, \mathbf{s}; w_{t-1}^i\right)$$

Note: The state $\mathbf{s}$ includes age.

Note: We are using the updated prices including $\{w_t^i\}_{t=1}^T$.

Note: We are holding fixed all future consumption and labor choices $\left\{c_q^i, n_q^i\right\}_{q=t}^T$ at the optimal value found in step (4), before having updated prices.

Because we don't let future consumption or labor react we are capturing only the change in savings due to the change in the returns to assets. Without taking into account the effect on resources in future periods.

(c) The problem above can be solved with the same EGM method used in our main algorithm:

i. Set an $a' \in \vec{a}$.

ii. If the labor choice is positive: $\tilde{n} > 0$, then current consumption solves:

$$\tilde{c} = \left(\frac{\gamma(1-\tau_l)\, w_{t-1}^i \hat{y}_h(\kappa, e)}{(1+\tau_c)(1-\gamma)}\right)^{\frac{-(1-\sigma)(1-\gamma)}{\sigma}}$$
$$\left[\beta s_h E_{e', z^{s'}}\left(\tilde{Y}_a\left(a', z'; \mathcal{R}_t^i, r_t^i\right)\left(c_t^i\left(\mathbf{s}'\right)\right)^{(1-\sigma)\gamma-1}\left(1 - n_t^i\left(\mathbf{s}'\right)\right)^{(1-\sigma)(1-\gamma)}\right)\right]^{\frac{-1}{\sigma}}$$

where $\mathbf{s}' = \left(h+1, a', z^p, \kappa, e', z^{s'}\right)$. This consumption implies a labor supply given by:

$$\tilde{n} = 1 - \frac{(1+\tau_c)(1-\gamma)}{\gamma(1-\tau_l)\, w_{t-1}^i \hat{y}_h(\kappa, e)}\tilde{c}$$

check that labor satisfies non-negative if so jump to (iv), if not go to (iii).

iii. If the labor choice is zero: $\tilde{n} = 0$, then current consumption solves:

$$\tilde{c} = \left[\beta s_h E_{e', z^{s'}}\left(\tilde{Y}_a\left(a', z'; \mathcal{R}_t^i, r_t^i\right)\left(c_t^i\left(\mathbf{s}'\right)\right)^{(1-\sigma)\gamma-1}\left(1 - n_t^i\left(\mathbf{s}'\right)\right)^{(1-\sigma)(1-\gamma)}\right)\right]^{\frac{1}{(1-\sigma)\gamma-1}}$$

where $\mathbf{s}' = \left(h+1, a', z^p, \kappa, e', z^{s'}\right)$.

iv. This implies an endogenous value of current non-labor income $\tilde{Y}\left(a'\right)$ given by:

$$\tilde{Y}\left(a^{'}, z^{p}, \kappa, e, z^{s}\right) = a^{'} + (1+\tau_c)\,\tilde{c} - Y^{n}\left(\tilde{n}, \mathbf{s}; w_{t-1}^{i}\right)$$

A. Obtain the the policy function for consumption by interpolating the values of $\tilde{c}_{t-1}(\mathbf{s})$, at known values of $\tilde{Y}$, for the values of $Y(a,z)$ implied by the grid $\vec{a}$. We use linear interpolation.

B. Obtain the policy function for labor from the first order condition as:

$$\tilde{n}_{t-1}(\mathbf{s}) = \max\left\{0\ ,\ 1 - \frac{(1+\tau_c)(1-\gamma)}{\gamma(1-\tau_l)\,w_{t-1}^{i}\hat{y}_h(\kappa,e)}\tilde{c}_{t-1}(\mathbf{s})\right\}$$

C. The value of $a_h^{'}(s)$ is:

$$\tilde{a}_{t-1}^{'}(s) = \tilde{Y}\left(a, z; \mathcal{R}_{t-1}^{i}, r_{t-1}^{i}\right) + Y^{n}\left(\tilde{n}_{t-1}(\mathbf{s}), \mathbf{s}; w_{t-1}^{i}\right) - (1+\tau_c)\,c_h(s)$$

v. Use the new auxiliary savings function $a_h^{'}(s)$ for updating the distribution of assets in period t $\left(\tilde{\Gamma}_t^{i}\right)$ taking as given the distribution in $t-1$ $\left(\Gamma_{t-1}^{i}\right)$. This step is only relevant for periods $t = 2, \ldots, T$ because the distribution in period $t = 1$ is fixed. The change in the distribution is not used for future dates.

We solve this until $\omega = 1$ and all the debt is absorbed by the local market.