Lecture 10: GE with Bells and Whistles

Fatih Guvenen

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  - Everybody trades corporate bonds.
  - A fraction μ of households also trade firm's stocks.
- The firm:
  - operates a Cobb-Douglas technology: $Y_t = Z_t K_t^{\theta} L_t^{1-\theta}$
  - faces capital adjustment costs: $K_{t+1} = (1-\delta) K_t + \Phi\left(\frac{I_t}{K_t}\right) K_t$
  - and finances χ fraction of its capital stock through debt.

Firm's Problem

- Firm's problem is dynamic:

$$P_t^s = \max_{\{I_{t+j}, L_{t+j}\}} E_t \left[\sum_{j=1}^{\infty} \beta^j \Lambda_{t,t+j} D_{t+j} \right]$$

$$\text{s.t.} \qquad K_{t+1} = (1-\delta) K_t + \Phi\left(\frac{I_t}{K_t}\right) K_t$$

where

- $D_t = Z_t K_t^{\theta} L_t^{1-\theta} - W_t L_t - I_t - \left(1 - P_t^f\right) \chi \overline{K},$
- $\Lambda_{t,t+j}$: IMRS of stockholders
- $\log(Z_{t+1}) = \phi \log(Z_t) + \varepsilon_{t+1}, \quad \varepsilon \overset{iid}{\sim} N(0, \sigma_\varepsilon^2).$

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- Here, the best choice turns out to be:
  - beginning of period capital stock, K
  - end-of-period bond holdings of bondholders, B
  - and technology shock, Z.

Stockholders and Non-Stockholders Problems

Stockholders:

$$V^h(\omega;\Upsilon) = \max_{c,l,b',s'} \left[(1-\beta)\, u(c, 1-l) + \beta \left(E\left[V^h(\omega';\Upsilon') \,|Z \right]^{1-\alpha^i} \right)^{\frac{1-\rho^i}{1-\alpha^i}} \right]^{\frac{1}{1-\rho^i}}$$

s.t.

$$c + P^f(\Upsilon) b' + P^s(\Upsilon) s' \le \omega + W(K, Z)\, l$$

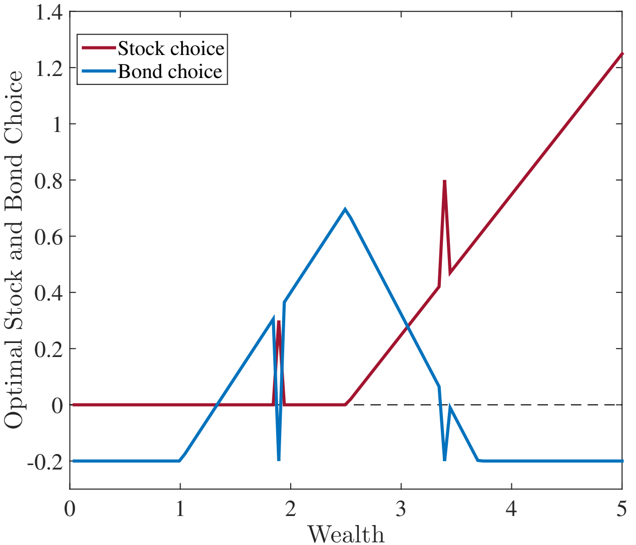
$$\omega' = b' + s'\left(P^s(\Upsilon') + D(\Upsilon')\right)$$

$$K' = \Gamma_K(\Upsilon),\ B' = \Gamma_B(\Upsilon)$$

$$b' \ge \underline{B},$$

Non-stockholders: set $s' \equiv 0$.

Portfolio Choice Problem: Caution!



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2. Optimal investment rule for the firm, $I(\Upsilon)$.
3. The equilibrium stock and bond pricing functions, $P^s(\Upsilon)$, and $P^f(\Upsilon)$.
4. The equilibrium laws of motion $\Gamma_K(\Upsilon)$, $\Gamma_B(\Upsilon)$ for the wealth distribution.

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4. Dealing with leverage (I will not talk about this today. See the paper for details.)
5. Gauss-Seidel vs Jacobi: ie., should you update the equilibrium functions in parallel in each round or successively? Makes a big difference on stability.

Issues Before Getting to the Algorithm

- Epstein-Zin preferences introduce important non-linearities into the firm's problem:

$$\Lambda_{t,t+1} = \beta^{\frac{1-\alpha}{1-\rho}} \left(\frac{c_{t+1}}{c_t}\right)^{-\rho} \left[\frac{V_{t+1}/c_t}{\left[\left(\frac{V_t}{c_t}\right)^{1-\rho} - (1-\beta)\right]^{\frac{1}{1-\rho}}}\right]^{\rho-\alpha},$$

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- When β is close to 1 (which is the case here: 0.996 because this is a monthly model) small deviations from the equilibrium Λ can yield large deviations in firm's behavior and thus in everything else.
- Solution: Pretend your are driving on ice. Make no sudden movements and you will be fine.

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  - Question: What should we take the initial CRRA coefficient? Is it easier to solve this model with low risk-aversion or high-risk aversion?
  - Not as obvious as it may seem!
- A two-agent model vs a model with infinitely many agents: Which one is harder to solve?
  - The answer is not obvious, because infinitely many agents often yields a degree of smoothness. That is, nobody matters much so pricing functions are not very sensitive.

Details of the Algorithm (1)

Step 0. Initialization:

(a) First construct grids. I used 25 points each for ω^h and ω^n; 5 and 20 points for K and B; and a 15-state Markov approx. to Z. Let i and j index grid points and iteration number respectively. All the steps below are performed for each point in the state space, Υ, and off-grid values are obtained by cubic spline interpolation.

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(b) Initial guesses for $\Gamma_K^0(\Upsilon)$, $\Gamma_B^0(\Upsilon)$, $P^{B,0}(\Upsilon)$, and $c^{h,0}(\omega;\Upsilon)$ are obtained by solving a simpler model where adjustment costs and leverage are eliminated and $\alpha = \rho$. To obtain $P^{s,0}(\Upsilon)$:

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  (i) First, solve the firm's investment problem using the initial guesses above. But stockholders' wealth, $\varpi^{h\prime} = ((P^{s\prime} + D') - B')/\mu$, needed to construct Λ, depends on $(P^s + D)$, which I do not have yet. Thus, I replace it with $(1 + R(K,Z))K$, where $R = r_t = \alpha Z_t (K/L)^{-\alpha}$.

Details of the Algorithm (2)

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(iii) Define and initialize the auxiliary variable $\overline{P}^m(K, B, Z) = K$ for $m = 0$. Now iterate on the mapping until $m = 50$.

$$\overline{P}^{m+1}(\Upsilon) = E\left[\beta\Lambda(\Upsilon, \Upsilon')\left(D^0(\Upsilon') + \overline{P}^m(\Upsilon')\right) \mid \Upsilon\right], \tag{1}$$

and set $P^{s,0}(\Upsilon) = \overline{P}^{50}(\Upsilon)$.

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(c) Now, taking the initial conditions above and setting $j = 1$, start the main iteration.

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(a) Solve the firm's investment problem as in (i)-(ii) above, using $c^{h,j}(\omega;\Upsilon)$ and $V^{h,j}(\omega;\Upsilon)$ obtained in step 1 to construct the discount factor. Note also that $\varpi^{h\prime} = ((P^{s'} + D') - B')/\mu$

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(b) Obtain $D^j(\Upsilon)$ using $I^j(\Upsilon)$.

(c) Obtain $\Gamma_K^j(\Upsilon)$ using $I^j(\Upsilon) : K'(\Upsilon) = (1-\delta)K + \Phi\left(I^j(\Upsilon)/K\right)K$.

(d) Obtain $P^{s,j}(\Upsilon)$ using the updated consumption and dividend decision rules via the recursion (1). Note that I set the initial condition $\overline{P}^0(\Upsilon) = P^{s,j-1}(\Upsilon)$ in (b-ii)

Details of the Algorithm (4)

Step 3: *Update the bond pricing function.* Find a bond price (at a given grid point Υ_i, in iteration j) which clears the bond market today when both agents take $P^{f,j-1}(\Upsilon)$, and (the newly updated) $P^{s,j}(\Upsilon)$ to apply in all future periods.

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i. Solve (and with $s' = 0$ for non-stockholders):

$$\max_{c,l,b',s'} \left[(1-\beta)\, u\,(c, 1-l) + \beta \left(E\left[V^h\left(\omega'; \Upsilon'\right)|Z\right]^{1-\alpha^i} \right)^{\frac{1-\rho^i}{1-\alpha^i}} \right]^{\frac{1}{1-\rho^i}}$$

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$$c + q b' + P^s(\Upsilon) s' \leq \omega + W\,(K, Z)\, l$$

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$$K' = \Gamma_K\left(\Upsilon\right), B' = \Gamma_B(\Upsilon)$$

$$b' \geq \underline{B},$$

Details of the Algorithm (5)

1. Obtain quasi-bond demand functions $\tilde{b}^h(\omega; \Upsilon, \hat{q})$ and $\tilde{b}^n(\omega; \Upsilon, \hat{q})$ as a function of the current bond price $\hat{q}$.

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2. Then, at each grid point Υ_i, search over the bond price $\hat{q}$ to find q_i^* such that the bond market clears:

$$\left|\mu\tilde{b}^h(\omega; \Upsilon, q_i^*) + (1-\mu)\,\tilde{b}^n(\omega; \Upsilon, q_i^*)\right| < 10^{-8}.$$

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- Trying to simultaneously clear both markets (and update the stock price as well) in this step creates instability in the algorithm and fails to converge. Instead, the iterative method described here (updating $P^{s,j}$ from (1), and $P^{f,j}$ from market clearing) works quite well in practice.

Details of the Algorithm (6)

Step 4: Obtain $\Gamma_B^j(\Upsilon)$: $\quad B'(\Upsilon) = (1-\mu)\,\tilde{b}^n\,(\varpi^n; \Upsilon, q^*(\Upsilon))$ where $\tilde{b}^n$ is non-stockholders' bond choice at the market clearing bond price in Step 3 (and not the one obtained in Step 1.)

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Finally, check stock market clearing $\mu s' = 1$ (which has not been imposed explicitly above).