Lecture 4: Miscellaneous Numerical Tools

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High-Quality Free Source Codes: Where to Find?

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  - Numerical Recipes code (Fortran and C)
  - GNU Scientific Library (C): http://www.gnu.org/software/gsl/
  - Netlib repository (Fortran and C): https://www.netlib.org
  - NLOPT: lots of optimization routines in many languages: https://nlopt.readthedocs.io/

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  - NLOPT: lots of optimization routines in many languages: https://nlopt.readthedocs.io/
- In general, avoid downloading software from some random researcher's website.
  - You often won't know who originally wrote the code,
  - Whether it was modified for the specific use of that researcher.

Maximizing RHS

$$V(k,z) = \max_{c,k'} \left[u(c) + \beta \int V(k',z') f(z'|z) dz' \right]$$

$$c + k' = (1+r)k + z$$

$$z' = \rho z + \eta.$$

Three Steps:

1. As part of maximizing the RHS, evaluate $\mathbb{E}(V(k',z')|z)$ *repeatedly*. Two components:
    1. Interpolation: Maybe required twice.
        1. Interpolate in the k' direction.
        2. Also interpolate in the z' direction, if $f(z'|z = z_j)$ is continuous.
    2. Integration: Again, if $f(z'|z = z_j)$ is continuous, we need to integrate. One option will be to treat it as discrete (saves time and headaches; not always feasible).

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2. How to perform the maximization in the Bellman objective? (Constrained optimization)

Root Finding

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Introduction

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  - Depending on f, there may be zero, one, or multiple solutions.
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- $N = 1$ is a special case where we can guarantee that we are bracketing a zero.
- In higher-dimensional problems, no method can (generically) guarantee that you are bracketing a zero.
- In higher dimensions, I often convert zero finding into a minimization problem. More on this in a moment.

Bisection, Secant and False Position Methods

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- Secant and False position methods:
  - Idea: Treat $f(\bullet)$ as if it is linear near the zero. Given $f(a)$ and $f(b)$, solve for the zero of the line that connects these two points.

Secant and False Position Methods

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- **False position**: take two most recent points that bracket zero.
  - Typically slower than Secant because it sometimes keeps an older evaluation in the interest of bracketing a zero (hard to determine exact convergence rate).

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- **TIP:** Convert "zero finding" into a minimization problem: if $f(a) \times f(b) < 0$ then the squared function $f(\bullet)^2$ will have a local minimum at the zero of $f(\bullet)$.
- Then one can use various methods for minimization (not surprisingly Brent and Newton's methods also have versions for optimization).

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- However, it carefully checks that the points always bracket a zero and the algorithm converges nicely. If not, it reverts back to bisection for a while.

Newton-Raphson Method

- First-order Taylor approximation:

$$f(x+\delta) = f(x) + \delta f'(x) + \text{higher order terms we will ignore...}$$

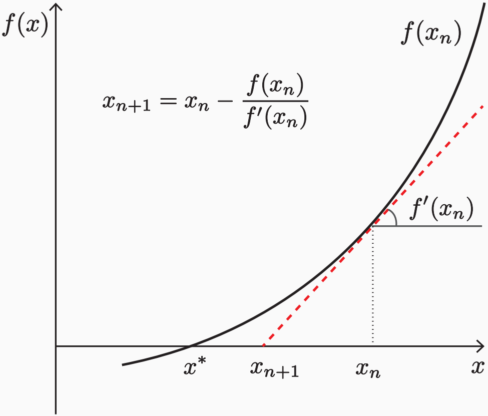
Setting $f(x+\delta) = 0$ yields $\delta = -\frac{f(x)}{f'(x)}$.

- Therefore, begin with x_0 and update:

$$\delta = x_{t+1} - x_t \Rightarrow x_{t+1} = x_t - \frac{f(x)}{f'(x)}$$

until convergence (if it does!)

Newton-Raphson method



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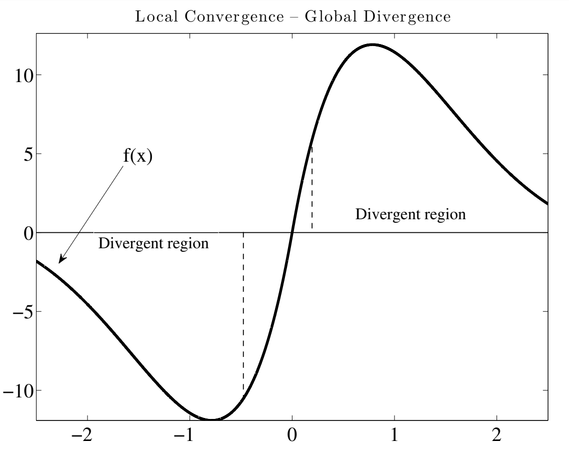
- **What to do?** Use a slow but sure method early on (like bisection) but then once you are “close” to the root switch to Newton-Raphson.

Poor Global Convergence

Figure 1: Global Divergence, Local Convergence



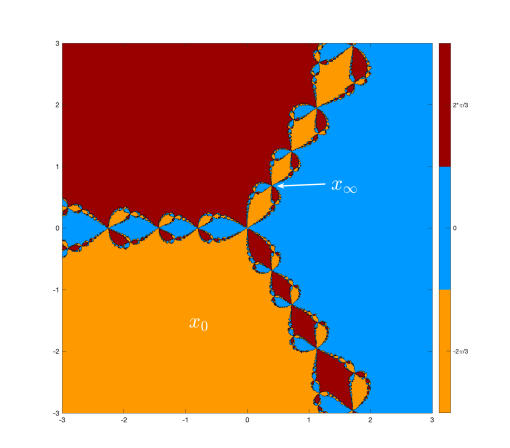
Newton's Fractals

- Consider the equation: $z^3 - 1 = 0$. It has three roots: $z_1 = 1$, $z_2 = -0.5 + i\sqrt{3}/2$, $z_2 = -0.5 - i\sqrt{3}/2$.
- The three points line up on a unit circle in the complex plane, separated by 120 degrees.
- Depending on the starting point, Newton's method can converge to one of three points or can bounce around with chaotic dynamics.
- The whole set feature fractals (near the boundaries).

Newton's Fractals

- Starting Newton's iteration to find the root of $z^3 - 1 = 0$ from an x_0 in a region with a given color, converges to one of the three roots of the same color.

Figure 2: Original set



Poor Global Convergence: Another Example

- Finding the zeroes of $p(x) = (x+3)(x-1)(x-4)$ using Newton's method starting from x_0 :

Table 1: Newton's method's limit

Initial point x_0	$\lim_{n\to\infty} x_n$
2.35287527	4
2.3528**4172**	−3
2.3528**3735**	4
2.35283**6327**	−3
2.35283632**3**	1

NUMERICAL DERIVATIVE

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- If $f(x)$ is excess demand and x is the price, calculating $f(x+h)$ and $f(x)$ requires solving the entire model twice!

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- If $f(x)$ is excess demand and x is the price, calculating $f(x+h)$ and $f(x)$ requires solving the entire model twice!
- If you have a complicated model this can be very time consuming.

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   - Because $\epsilon_f \geq \epsilon_m$ (machine precision: typically ~10^{-15} for DP), the best lower bound is $\sim \sqrt{\epsilon_m}$.
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   - Notice that in some cases taking a small h may amplify the truncation error leading to **less** precision in f'.
   - If you choose a termination condition for your program that is too small (e.g., $f' < \epsilon_f/2$), it may never converge!

Numerical Differentiation

- Two-Sided Derivatives: A better but slower approach:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x-h)}{2h}$$

- This also requires two function evaluations. So why is it slower?
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  - So two-sided derivatives require 2 additional evaluations compared to 1 for one-sided.
- What's the benefit?
  - For one-sided derivatives, truncation error is $\sim h$. For two sided, it is $\sim h^2$!
  - Choose a smaller h than in one-sided case though: assuming you know the error in evaluating $f(x)$, choose $h \sim \epsilon_f^{1/3}$.