Lecture 9: GE with Heterogeneity and Aggregate Shocks

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Introduction

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- **Q:** What happens when we introduce an aggregate shock (e.g., a TFP shock) to an Aiyagari (1994)-style model?
  - Now, equilibrium prices are no longer constant.
  - This is because the wealth distribution (call μ) will vary with the aggregate shock (call Z), so market clearing prices will also vary.
  - In other words, equilibrium pricing functions depend on Z **and** the entire wealth distribution!

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  - and how μ evolves: $\mu_{t+1} = \Gamma(\mu_t, Z_t)$
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  - Without any further discipline, this is an impossible task!
- So, many thought this was an intractable problem, which blocked the way to business cycle analysis with incomplete markets.

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- We will cover two papers that solve these three problems:
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- There are other contemporaneous contributions by Den Haan (1996) and Rios-Rull (1997) as well.

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  - Expanding the exponential:
    $M_X(t) = 1 + tE(X) + \frac{t^2 E(X^2)}{2!} + \frac{t^3 E(X^3)}{3!} + \ldots + \frac{t^n E(X^n)}{n!} + \ldots$

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- Assume ϵ is *i.i.d* conditional on $z \Rightarrow$ fraction of employed (hence ℓ) only depends on z.
- Competitive markets: $w(K, L, z) = (1-\alpha)z(K/L)^{-\alpha}$ and $r(K, L, z) = \alpha z(K/L)^{\alpha}$

$$V(k, \epsilon; \mu, z) = \max_{c,k'} \left[u(c) + \beta E[V(k', \epsilon'; \mu', z')] | z, \epsilon\right]$$

$$c + k' = w(K, L, z) \times \ell \times \epsilon + r(K, L, z) \times k, \qquad k' \geq 0$$

$$\mu' = \Gamma(\mu, z, z')$$

Krusell-Smith Algorithm

1. Approximate μ_k with a finite number of moments: $\mathbf{m} \equiv (m_i)_{i=1}^{N}$. The mapping $\Gamma(.,Z)$ reduces to a vector-valued function: $\mathbf{m}' = \bar{\Gamma}(\mathbf{m}, z) : \mathbb{R}^N \to \mathbb{R}^N$. Select a parametric family of functions for Γ. A linear or log-linear function is a common choice: $\mathbf{m}'(z) = \mathbf{A}_0 + \mathbf{A}_1(z)\mathbf{m}$.

2. Make an (educated) initial guess about $(\mu_0, \mathbf{A}_0, \mathbf{A}_1)$.

3. Solve the individual's dynamic program (*P2*):

$$V(k, \epsilon; \mathbf{m}, z) = \max_{c,k'} \left[U(c) + \beta \mathbb{E}[V(k', \epsilon'; \mathbf{m}', z') | \, z, \epsilon]\right] \tag{1}$$

$$c + k' = R(K, L, z) \times k + W(K, L, z) \times \bar{\ell} \times \epsilon, \tag{2}$$

$$\mathbf{m}'(z_j) = \mathbf{A}_0(z_j) + \mathbf{A}_1(z_j)\mathbf{m} \qquad \text{for } j = b, g, \tag{3}$$

$$k' \geq 0.$$

Krusell-Smith Algorithm (cont'd)

4. Using the resulting decision rules only, simulate $\left\{\tilde{k}_{n,t}\right\}_{n=1,t=1}^{N,T}$ for (N,T) large. Compute $\{\tilde{\mathbf{m}}_t\}_{t=1}^{T}$.

5. Discard the first $T^{\text{burn-in}}$ periods of simulated data $(0 \ll T^{\text{burn-in}} \ll T)$. Using the rest, estimate $(\hat{\mathbf{A}}_0, \hat{\mathbf{A}}_1)$ by running the following regressions:

$$\tilde{\mathbf{m}}'(z_j) = \hat{\mathbf{A}}_0(z_j) + \hat{\mathbf{A}}_1(z_j)\tilde{\mathbf{m}} + \nu \qquad \text{for } j = b, g, \tag{4}$$

   where $\widetilde{K} \equiv \frac{1}{N}\sum_{n=1}^{N}\tilde{k}_n$, and $\nu_i, i = b, g$, denote the residuals.

6. Iterate on steps 3 to 5 until the R^2 of the regression in (4) satisfies $R^2 > 1 - \epsilon_{R^2}$ and the forecast variance satisfies $\sigma_\nu < \underline{\sigma}$ for very small values of ϵ_{R^2} and $\underline{\sigma}$. If accuracy remains insufficient, go back to steps 1–2 and increase *M* or choose a different family of functions for Γ.

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- If you solve for the full recursive equilibrium you would not need to do this.
- However, solving for the full equilibrium is often much harder and so is often "over-kill."

Krusell-Smith: Details

1. As with many numerical methods, a good initial guess is critical.
   - One idea (which KS used) is to first solve a rep-agent RBC model with same parameterization as KS model. Then use its coefficients (a_0, a_1, b_0, b_1) as initial guess.
   - We will see some more complex examples next week.

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2. Can't we update H without simulating? Yes we can.
   - Den Haan and Rendahl (2009) propose a method where

     $$K^{'} = H_j(K, z) = \int k_j^{'}(k, \epsilon; \Gamma, z) d\Lambda(k, \epsilon)$$

     where $\Lambda(k, \epsilon)$ is the distribution of households across capital and employment status
   - This works well if you choose a parameterize the decision rules in particular way.

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- See "Solving the Incomplete Markets Model with Aggregate Uncertainty using Explicit Aggregation" on Wouter Den Haan's web site.

Accuracy of the Solution

Final solution in Krusell-Smith's baseline model:

$$\log \widetilde{K}' = 0.085 + 0.965 \log \widetilde{K} \quad \text{for } Z = Z_b, \quad R^2 = 0.999998, \;\; \sigma_\nu^2 = 0.0036\% \tag{5}$$

$$\log \widetilde{K}' = 0.095 + 0.962 \log \widetilde{K} \quad \text{for } Z = Z_g \quad R^2 = 0.999998, \;\; \sigma_\nu^2 = 0.0028\% \tag{6}$$

- Notice how high the R^2 is!
- Can you get away with something lower? Anything below $R^2 = 0.999$ is typically not an accurate solution. (More on this in a moment).

Substantive Results: A Digression

How Much Wealth Inequality Does the K-S Model Generate?

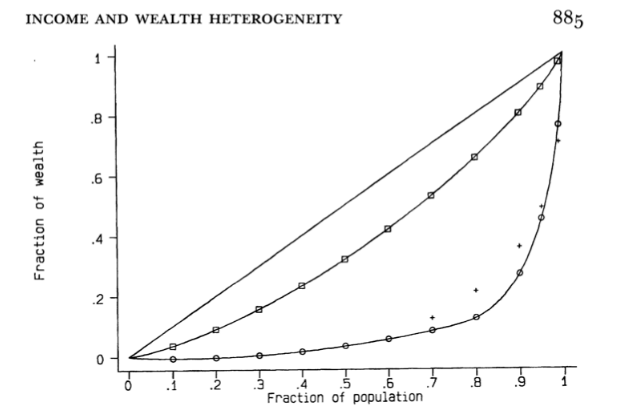

FIG. 3.—Lorenz curves for wealth holdings (+ refers to the data, □ to the benchmark model, and ○ to the stochastic-β model).

- Baseline model: Not much inequality. Gini is 0.25, top 1% holds 3% of wealth.

Stochastic-Beta Model

- To generate more wealth inequality, K-S introduce stochastic-β or patience heterogeneity:
  - $\widetilde{\beta}_t$: Markov chain with persistence matched to length of a generation.
- How well does the Krusell-Smith method work?

$$\log \widetilde{K}' = 0.095 + 0.961 \log \widetilde{K} \quad \text{for } Z = Z_b, \quad R^2 = 0.999985, \ \sigma = 0.0077\%$$

$$\log \widetilde{K}' = 0.100 + 0.961 \log \widetilde{K} \quad \text{for } Z = Z_g \quad R^2 = 0.999991, \ \sigma_u^2 = 0.0056\%,$$

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- How much wealth inequality does stochastic-β generate? A lot!

TABLE 1

DISTRIBUTION OF WEALTH: MODELS AND DATA

Model	Percentage of Wealth Held by Top 1%	5%	10%	20%	30%	Fraction with Wealth < 0	Gini Coefficient
Benchmark model	3	11	19	35	46	0	.25
Stochastic-β model	24	55	73	88	92	11	.82
Data	30	51	64	79	88	11	.79

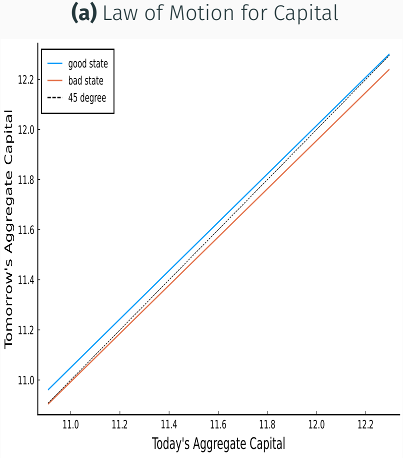
TABLE 2

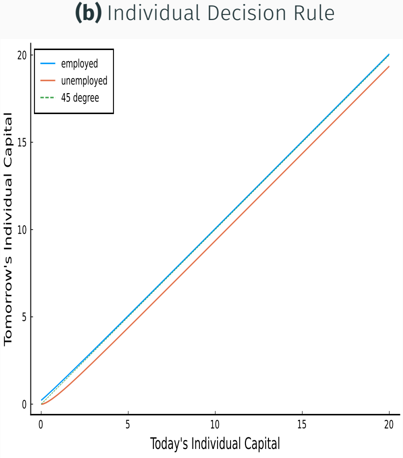
AGGREGATE TIME SERIES

Model	Mean(k_t)	Corr(c_t, y_t)	Standard Deviation (i_t)	Corr(y_t, y_{t-4})
Benchmark:				
Complete markets	11.54	.691	.031	.486
Incomplete markets	11.61	.701	.030	.481
$\sigma = 5$:				
Complete markets	11.55	.725	.034	.551
Incomplete markets	12.32	.741	.033	.524
Real business cycle:				
Complete markets	11.56	.639	.027	.342
Incomplete markets	11.58	.669	.027	.339
Stochastic-β:				
Incomplete markets	11.78	.825	.027	.459

How do the Decision Rules Look Like?

Figure 1: Solution to Krusell-Smith Model





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- A major reason for outcome is that savings rule is approximately linear in assets.
- So redistributing wealth from one agent to another would almost not affect aggregate savings.
- This depends on a few key assumptions:
  - identical preferences (Rubinstein showed that with such preferences it's easy to get full demand aggregation).
  - single asset (will talk about multiple assets in next lecture).
  - big aversion to constraints (zero labor income makes it very costly to be at constraint).
  - few people at the constraint and even if there are, they have no wealth to affect prices.

(Nonlinear) Effect of Income Persistence on Inequality

Table 1. Effect of increased idiosyncratic income persistence

	ρ			
Variable	0.50	0.98	0.99	1.00
Equilibrium interest rate, %	4.12	4.07	4.06	4.07
Aggregate capital	11.60	11.67	11.68	11.66
SD of capital	1.47	6.42	5.94	0.36
Skewness of capital	–0.03	2.58	3.60	4.98
Gini coefficient of capital	0.067	0.255	0.217	0.018

- Effect of ρ nonlinear at the top: look at $\sigma(\mathrm{K})$ and Gini of wealth.

Back to Computation:

Checking Accuracy of Solution

Since Krusell-Smith (1998)

- Approximate aggregation has been found to hold in a wide range of GE models with heterogeneity:
  - ? reports that adding higher moments added little precision. ? follows Midrigan and does something similar.
  - Khan and Thomas (AER, 2007; ECMA, 2008) get good results using the mean only.
  - Krusell, Mukuyama, Sahin (labor search application)
  - Bloom (ECMA, 2009), Bloom et al (ECMA, 2018): Rich GE firm dynamics model with uncertainty shocks: K-S still works.
  - Many many more..

JEDC 2010 Special Issue on Krusell-Smith: Wide Variation in Speed and Accuracy of Algorithms

Figure 2: Different Algorithms for Solving Krusell-Smith (1998)

Table 2: Computation times

algorithm	programming language	time	authors
BInduc	Matlab	47 minutes	Michael Reiter
KS-num	Fortran	324 minutes	Lilia Maliar, Sergui Maliar, Fernando Valli
KS-sim	Matlab	310 minutes	Eric Young
Param	Fortran	2739 minutes	Yann Algan, Olivier Allais, Wouter den Haan
Xpa	Matlab	7 minutes	Wouter den Haan, Pontus Rendahl
Penal	Matlab	$<$ 1 second!	Henry Kim, Robert Kollmann, Jinill Kim

Notes: This table reports the time it takes to solve the model when $\gamma = 1.1$, starting at the solution for $\gamma = 1$.

Checking for Accuracy of your Solution

- Den Haan's experiment: Suppose the true solution to K-S model is given by:

$$\log \mathrm{K}' = \alpha_1 + \alpha_2 \mathrm{z} + \alpha_3 \log \mathrm{K}$$

  with parameters in the first row of table below.

- Change α_3 such that the R^2 gradually falls to values shown in the following rows while keeping mean K constant.
- Check what happens to implied standard deviation of capital (last column):

Table 1: Meaninglessness of the R^2

			implied properties	
equation	R^2	$\widehat{\sigma}_u$	mean	stand. dev.
$\alpha_3 = 0.96404$ (fitted regression)	0.99999729	4.1×10^{-5}	3.6723	0.0248
$\alpha_3 = 0.954187$	0.99990000	2.5×10^{-4}	3.6723	0.0217
$\alpha_3 = 0.9324788$	0.99900000	7.9×10^{-4}	3.6723	0.0174
$\alpha_3 = 0.8640985$	0.99000000	2.5×10^{-3}	3.6723	0.0113

Notes: The first row corresponds to the fitted regression equation. The subsequent rows are based on aggregate laws of motion in which the value of α_3 is changed until the indicated level of the R^2 is obtained. α_1 is adjusted to keep the fitted mean capital stock equal.

True DGP: $\log \mathrm{K}' = \alpha_0 + \alpha_1 \log \mathrm{K} + \alpha_2 \mathrm{z} + \alpha_3 \log \mathrm{K}_{-1}$

Approximated DGP: $\log \mathrm{K}' = \alpha_0 + \alpha_1 \log \mathrm{K} + \alpha_2 \mathrm{z}$

- Experiment 2: $\log \mathrm{K}' = \alpha_0 + \alpha_{1,\mathrm{t}} \mathrm{logK} + \alpha_2 \mathrm{z}$. Ignore time variation in α_1.

Table 2: Parameter Values

Parameter	Experiment #1		Experiment #2	
	#1.1	#1.2	#2.1	#2.2
α_0	0	0		0
α_1	1.08	1.38	0.65	0.65
α_2	1	1	1	1
$\alpha 3$	-0.1	-0.4	0.3	0.3
α_4	-	-	0.01	0.01
α_5	-	-	50	50
ρ_0	0	0	0	0
ρ_1	0	0	0.95	0
σ	0.00472	0.15436	$6.3891 * 10^{-4}$	$8.616 * 10^{-3}$

Notes: All parameter sets imply a standard deviation for the underlying series equal to 2.5%.

Checking Accuracy

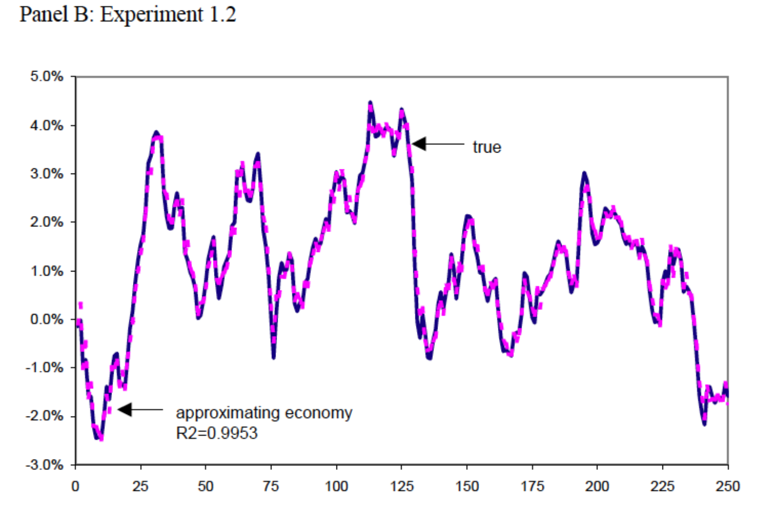
Table 3: Traditional Accuracy Tests

	Experiment #1.1		Experiment #1.2	
T	3000	50000	3000	50000
average R^2 (level)	0.9996	0.9996	0.9952	0.9955
minimum R^2 (level)	0.9995	0.9996	0.9940	0.9951
average R^2 (Δ)	0.9901	0.9901	0.8413	0.8411
minimum R^2 (Δ)	0.9901	0.9901	0.8408	0.8410
average $\widehat{\sigma}_u$	0.047%	0.047%	0.168%	0.168%
maximum $\widehat{\sigma}_u$	0.049%	0.048%	0.174%	0.170%
	Experiment #2.1		Experiment #2.2	
T	3000	50000	3000	50000
average R^2 (level)	0.99993	0.99993	0.99986	0.99986
minimum R^2 (level)	0.99983	0.99991	0.99971	0.99982
average R^2 (Δ)	0.97695	0.97559	0.99879	0.99880
minimum R^2 (Δ)	0.93847	0.96828	0.99750	0.99847
average $\widehat{\sigma}_u$	0.021%	0.022%	0.030%	0.030%
maximum $\widehat{\sigma}_u$	0.034%	0.025%	0.044%	0.034%

Notes: The standard deviation of the true series is equal to 2.5%. R^2 for the "level" (Δ) regression is based on a regression with m_{t+1} ($m_{t+1} - m_t$) as the dependent variable.

Essential Accuracy Plot

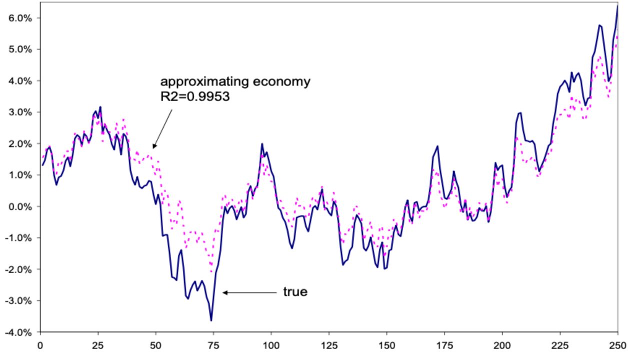
Figure 3: Updated Values of K used in Approximating Law



Essential Accuracy Plot

Figure 4: Updated Values of K not used in Approximating Law

Figure 4: High R^2 and inaccurate law of motion



Notes: This graph plots the true aggregate capital stock and the one predicted by the approximate aggregate law of motion *when* the input of the approximation is the lagged value generated by the approximation *not* the true lagged value (as is done when calculating the R^2.

Rules of Thumb

1. Do not rely on R^2 and residual variance alone.
   - These are average measures. Can hide big inaccuracies confined to small areas that can be very important.
   - R^2 is scaled by the LHS of the regression. An alternative is to check the R^2 of:

$$\log \mathrm{K}' - \log \mathrm{K} = \alpha_1 + \alpha_2 \mathrm{z} + (\alpha_3 - 1) \log \mathrm{K}$$

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   - Krusell and Smith checked 25 years ahead (100 periods)
3. Den Haan suggests comparing the law of motion generated by simulated data by plotting the "essential accuracy plot" shown above.

Wide Variation in Speed of Algorithms

Figure 5: Different Algorithms for Solving Krusell-Smith (1998)

Table 2: Computation times

algorithm	programming language	time	authors
BInduc	Matlab	47 minutes	Michael Reiter
KS-num	Fortran	324 minutes	Lilia Maliar, Sergui Maliar, Fernando Valli
KS-sim	Matlab	310 minutes	Eric Young
Param	Fortran	2739 minutes	Yann Algan, Olivier Allais, Wouter den Haan
Xpa	Matlab	7 minutes	Wouter den Haan, Pontus Rendahl
Penal	Matlab	$<$ 1 second!	Henry Kim, Robert Kollmann, Jinill Kim

Notes: This table reports the time it takes to solve the model when $\gamma = 1.1$, starting at the solution for $\gamma = 1$.

Checking for Accuracy: Christiano and Fisher (JEDC, 2000)

- Solve a representative agent asset pricing model for 7 different parameterizations and compare the accuracy of several solution algorithms

Christiano and Fisher: Solution Methods

Table 1
Summary of the computational strategies considered

Computational strategy[a]	Object approximated	Residual weighting scheme	Evaluation of integrals
Spectral methods[b]			
Conventional PEA	Marcet conditional expectation	Model-implied density for capital and technology	Monte Carlo
Modified conventional PEA	Marcet conditional expectation	Exogenous density for capital and technology	Monte Carlo
Chebyshev PEA	Marcet and Wright-Williams conditional expectation	Dirac delta functions (if collocation) Galerkin (if Galerkin)	Quadrature
PEA collocation	Marcet and Wright-Williams conditional expectation	Dirac delta functions	Quadrature
PEA Galerkin	Marcet conditional expectation	Galerkin	Quadrature
Spectral-Galerkin	Policy and multiplier functions	Galerkin	Quadrature
Finite element methods[c]			
FEM collocation	Policy and multiplier functions	Dirac delta functions	Quadrature
FEM Galerkin	Policy function	Galerkin	Quadrature

[a]These names are intended as a convenient shorthand only. For example, technically PEA Galerkin is a Spectral Galerkin method too.
[b]We used polynomials.
[c]We used piecewise linear functions.

Christiano and Fisher: Model Parameterizations

Table 2
Parameterizations considered

Model	Parameter values					
	β	γ	α	δ	σ	ρ
(1)	$1.03^{1/4}$	1	0.3	0.02	0.23	0
(2)	$1.03^{1/4}$	10	0.3	0.02	0.23	0
(3)	$1.03^{1/4}$	1	0.05	0.02	0.0382	0
(4)	$1.03^{1/4}$	1	0.3	0.5	0.675	0
(5)	$1.03^{1/4}$	1	0.3	0.02	0.23	0.95
(6)	$1.03^{1/4}$	1	0.3	0.02	0.40	0
(7)	$1.03^{1/4}$	10	0.1	0.02	0.23	0.95

Accuracy for Quantities: OK

L.J. Christiano, J.D.M. Fisher / Journal of Economic Dynamics & Control 24 (2000) 1179–1232 1207

Table 3
Bias and Monte Carlo variation in conventional PEA

	Parameterizations						
Statistic	(1)	(2)	(3)	(4)	(5)	(6)	(7)
Panel A – Quantities							
σ_y	66.0	67.9	3.95	68.8	84.8	125.0	34.2
	[− 0.2]	[− 0.6]	[− 0.04]	[− 0.3]	[0.1]	[− 0.3]	[− 0.2]
	(0.004)	(0.02)	(0.001)	(0.01)	(0.02)	(0.003)	(0.04)
	⟨0.1⟩	⟨0.5⟩	⟨0.01⟩	⟨0.1⟩	⟨0.5⟩	⟨0.1⟩	⟨0.9⟩
σ_c	10.2	7.50	1.01	34.3	49.7	45.4	12.4
	[− 1.1]	[1.1]	[− 0.2]	[− 0.9]	[− 0.2]	[− 1.0]	[6.3]
	(0.03)	(0.1)	(0.04)	(0.01)	(0.02)	(0.01)	(0.5)
	⟨0.7⟩	⟨1.6⟩	⟨0.9⟩	⟨0.3⟩	⟨0.0⟩	⟨0.3⟩	⟨10.6⟩
σ_i	61.9	65.9	3.42	36.1	43.8	80.8	23.1
	[0.04]	[− 0.8]	[0.3]	[− 0.1]	[0.3]	[0.2]	[− 3.8]
	(0.01)	(0.03)	(0.01)	(0.02)	(0.1)	(0.01)	(0.3)
	⟨0.2⟩	⟨0.6⟩	⟨0.2⟩	⟨0.4⟩	⟨1.4⟩	⟨0.3⟩	⟨7.3⟩

Asset Prices more Sensitive than Quantities

Panel B – Asset prices and returns

ER^e	3.20	3.08	3.01	309.5	2.94	59.8	1.44
	[− 0.1]	[25.1]	[0.4]	[2.0]	[− 0.2]	[− 1.7]	[1.6e 8]
	(0.05)	(1.9)	(0.02)	(0.1)	(0.1)	(0.04)	(1.6e 8)
	⟨1.04⟩	⟨33.6⟩	⟨0.5⟩	⟨1.1⟩	⟨2.7⟩	⟨0.9⟩	⟨2124⟩
ER^f	3.00	2.47	3.00	8.6	2.88	19.7	− 5.42
	[1.8]	[9.0]	[0.6]	[30.3]	[− 0.1]	[2.6]	[36.8]
	(0.01)	(0.3)	(0.001)	(0.1)	(0.01)	(0.03)	(2.7)
	⟨0.13⟩	⟨6.3⟩	⟨0.01⟩	⟨1.4⟩	⟨0.2⟩	⟨0.6⟩	⟨42.3⟩
$E(R^e - R^f)$	0.20	0.60	0.02	300.9	0.06	40.1	6.86
	[− 25.8]	[93.5]	[− 77.6]	[1.2]	[− 3.5]	[− 3.8]	[3.3e 7]
	(0.7)	(8.5)	(4.6)	(0.1)	(6.1)	(0.04)	(3.3e 7)
	⟨20.4⟩	⟨97.9⟩	⟨460.7⟩	⟨1.1⟩	⟨141.3⟩	⟨1.0⟩	⟨2124⟩

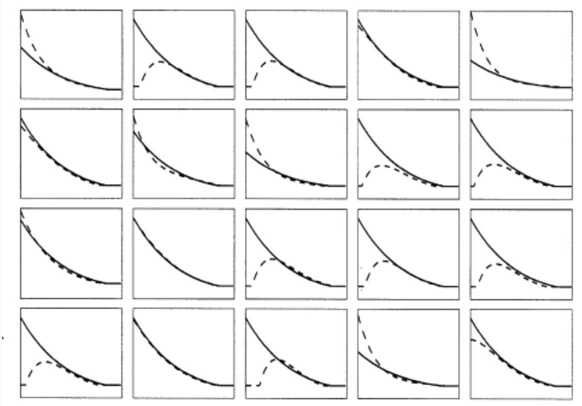
Asset Prices more Sensitive than Quantities

Panel B – Asset prices and returns

ER^e	[1.6e 8] (1.6e 8) ⟨2124⟩	[4.7e 7] (4.6e 7) ⟨687⟩	[− 13.0] (8.1) ⟨65.8⟩	[− 7.5] (24.2) ⟨159⟩	[− 29.8]	[− 8.6]
ER^f	[36.8] (2.7) ⟨42.3⟩	[4.2] (1.7) ⟨11.5⟩	[− 1.7] (− 1.6) ⟨11.8⟩	[− 0.8] (5.0) ⟨35.4⟩	[− 4.2]	[− 0.4]
$E(R^e - R^f)$	[3.3e 7] (3.3e 7) ⟨2124⟩	[9.9e 6] (9.6 e 6) ⟨687⟩	[− 4.1] (2.9) ⟨21.1⟩	[1.0] (8.7) ⟨61.1⟩	[− 9.6]	[− 2.1]
σ_q	[7.9] (1.1) ⟨22.0⟩	[− 1.9] (0.9) ⟨6.6⟩	[− 2.7] (1.2) ⟨8.8⟩	[− 5.0] (3.3) ⟨24.6⟩	[− 4.9]	[− 1.2]
$\rho(y, q)$	[− 0.3] (0.7) ⟨14.5⟩	[− 1.0] (0.9) ⟨6.4⟩	[− 1.9] (1.0) ⟨6.9⟩	[− 3.5] (1.8) ⟨13.3⟩	[− 0.8]	[− 1.2]
Freq$(q < 1)$	[− 7.2] (0.6) ⟨13.3⟩	[− 4.6] (0.9) ⟨6.8⟩	[− 2.0] (1.0) ⟨6.9⟩	[− 3.9] (1.8) ⟨12.9⟩	[0.1]	[− 1.1]

Understanding the Source of the Problem

Figure 6: Exact vs Numerically Computed Savings Decision Rules (for Different State Variable Values)



Back to Approximate Aggregation:

When Does it Hold/Not Hold?

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- In fact, there is a broad set of GE models with lots of bells and whistles where Krusell-Smith method still works.

Krusell-Smith: When Does it Work?

	Approx. Agg. Obtains?	K-S works?	What moments sued
Krusell-Smith (1997, MD): 2 asset problem	yes	yes	$\overline{\mathrm{K}}$, but nonlinear bond price eq.
Krusell-Smith (1998, JPE)	yes	yes	$\overline{\mathrm{K}}$
?, IER	yes	yes	
?, ECMA	yes	yes	
?, ECMA	yes	yes	
?, RED	yes	yes	
?, JME	yes	yes	
?, QJE	yes	yes	
?, ECMA	yes	yes	
Bloom et al (ECMA, 2018)	yes	yes	
?, RED	no	yes	$\overline{\mathrm{K}}$, cond. exp. equity premium
Guvenen (2001, JME)	no*	no	2 pt wealth distr needed. Full RCE solved +
Guvenen (2009, ECMA)	no*	no	*Eq. functions very nonlinear*
Kubler and Schmedders (2002, MD)	no	no	Wealth distr. is not sufficient state var.
Krueger and Kubler (2004, JEDC)	no	no	Wealth distr. is not sufficient state var.

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- Broadly speaking, two types of features make approximate aggregation not hold well.
- OLG structure (e.g., Krueger and Kubler (2004), ?):
  - because of shorter horizon and young starting with low wealth, many individuals in the nonlinear part of their savings decision rule.
- Aggregate shocks with redistributional structure (e.g., Guvenen (2001, 2009) and others with capitalist-entrepreneur/worker models):
  - Aggregate shocks have multiple effects, some affecting groups in same direction (through wage effect), others in opposite directions (different exposures to prices movements).
  - Standard GE models like K-S also have this feature but it is much more muted.

Example: Asset Pricing Model with Segmented Markets

Guvenen (2009, ECMA):

- Production economy with two assets: Firm's capital (stock) and a household bond.
- Two types of agents: stockholders (h) and bondholders (n). The latter cannot hold the stock.
- Population share of stockholders is μ, and share of bondholders is $1-\mu$.

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- (K, B) represents the entire wealth distribution since there are two agents.
- Both agents face portfolio constraints: $\mathrm{s}_{\mathrm{h}}' \geq 0$ and $\mathrm{b}_{\mathrm{h}} \geq \underline{\mathrm{b}_1}$, and $\mathrm{b}_{\mathrm{n}} \geq \underline{\mathrm{b}_2}$.

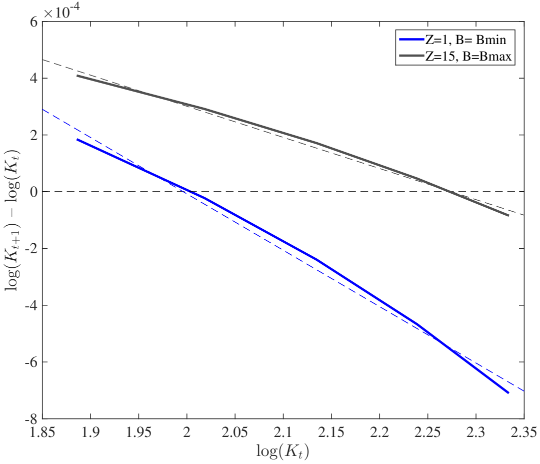
- So in equilibrium, stockholders as a group cannot hold more bond than what non-stockholders can borrow:

- So in equilibrium, stockholders as a group cannot hold more bond than what non-stockholders can borrow:
- In other words, since bond market clearing gives: $\mu b_h(K, B, Z) + (1 - \mu) b_n(K, B, Z) = 0$, for all aggregate states (K, B_{min}, Z) where $B_{\text{min}} = (1 - \mu)\underline{b_2}$ we must have $b_h(K, B_{\text{min}}, Z) = (1 - \mu)\underline{b_2}/\mu$ must be an optimal choice.
- Asset prices adjust near this constraint—sometimes very nonlinearly—to make this possible.

Law of Motion for Capital

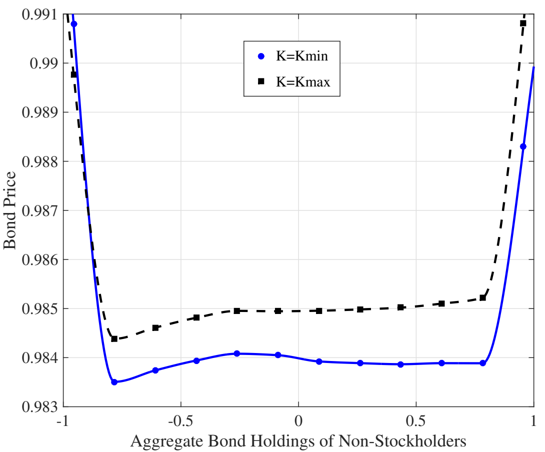
- Law of motion for capital has fairly low curvature in K direction.

Figure 7: Evolution of Aggregate (Log) Capital



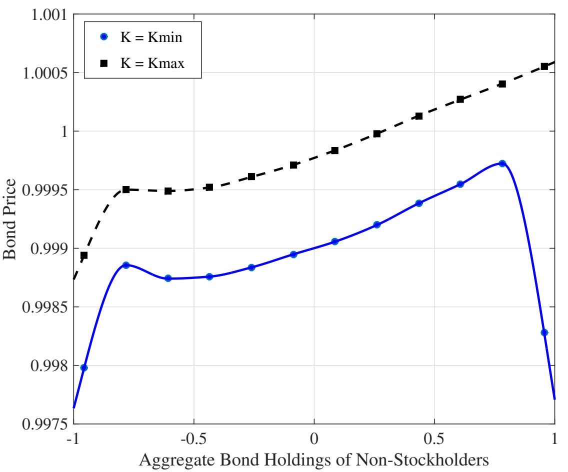
Equilibrium Bond Price Function, $Z = Z_{min}$

- Significant curvature near grid limits (borrowing limit of each group).



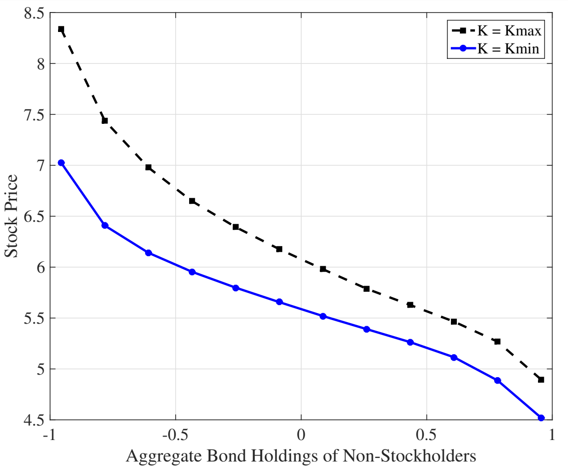
Equilibrium Bond Price Function, $Z = Z_{median}$

- Direction and shape of curvature changes significantly with level of (K, Z)



Equilibrium Stock Price Function, $Z = Z_{max}$

- Less curvature for high Z. Not evidence that there is little curvature for other (Z, K) combinations.



An aside

- A different and deeper question is whether the wealth distribution itself is a sufficient statistic to write a sequential GE problem in recursive form.
- In general: no proof that a recursive equilibrium exists with wealth as a state variable.
- A recursive equilibrium exists using an expanded state vector: Jianjun Miao (2006), Competitive Equilibria of Economies with a Continuum of Consumers and Aggregate Shocks, *JET*, 128, 274-298.

References