Lecture 1: Introduction and Model Specification

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Introduction

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1. **Mathematical/analytical/theory skills:**
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2. **Model building skills:**
   - Sensible and suitable choices for the problem at hand.
   - Captures key interactions without creating a monster.
   - Knows where to simplify and where to expand.

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4. Empirical skills:
   1. Low/Mid level: Knowing enough econometrics to do basic analysis correctly.
      - Solid knowledge of Stata/R.
   2. Mastery/Advanced: Doing state-of-the-art estimation of large-scale structural models using indirect inference; or doing causal inference, etc.
      - Coding in Stata/R/C++, replacing built-in Stata/R code with yours, etc.

Welcome to 2024: Now You Need a Fifth Skill!

- How to use ChatGPT, Claude, Grok, and other AI tools that are emerging for your research?

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- If you want to see the power, try this. Go to chat.openai.com and enter a query:
  - "Can you write a Python code for me to solve the Krusell Smith (1998, JPE) model?"
  - "Can you fill in the missing parts so that I can run this code and get a real solution?"

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Know Thyself: What Is Your Comparative Advantage?

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- Nobody can be—or has to be—great at everything. You just need to figure out what your comparative advantage is. And figure it soon.
- Then invest heavily in those skills. Especially in your 2nd and 3rd years.
- This course is focused on Skill 4: Empirical Analysis.

Model Specification

Individual Decision Problem

Model Specification: Individual Decision Problem

$$V(a,w) = \max_{c,\ell,a'} \left[u(c,\ell) + \beta \mathbb{E}(V(a',w')|w)\right]$$
$$c + a' = (1+r)a + w(1-\ell)$$
$$w' \sim f(\bullet|w)$$

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- How to specify $f(\bullet|w)$? The entire *income dynamics* literature is concerned with the choice of $f()$.
- How about if we have other shocks (health, wealth or rate-of-return, preferences, etc.)?

Specification 1:

$$V(a,w) = \max_{c,\ell,a'} \left[\left(\frac{c^{1-\sigma}}{1-\sigma} + \psi \times \frac{\ell^{1-\gamma}}{1-\gamma} \right) + \beta \mathbb{E}(V(a',w')|w) \right]$$

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3. How about $\ell^{i*}(a,w) = 1$? Yes.
   - NB: No reason $\ell^{i*} > 1$ cannot be optimal but we rule it out by assuming $\ell^i \leq 1$, so wage income not negative.

Preferences: How About Now?

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- Can $\ell^{i*}(a,w) = 0$ ever be optimal? YES (Why?)
- How about $\ell^{i*}(a,w) = 1$? Yes, if:
  - $U_\ell > w\lambda$ where λ is marginal utility of wealth. So high income/wealth effect $\Rightarrow$ enjoy leisure full time.

Planning Problem

Similar yet Different: Planning Problem

$$\max_{\{C_t, N_t K_{t+1}\}} \quad \sum_{t=0}^{\infty} \beta^t \left[\frac{C_t^{1-\sigma}}{1-\sigma} + \psi \times \frac{(1-N_t)^{1-\gamma}}{1-\gamma} \right]$$

$$s.t. \qquad C_t + K_{t+1} - (1-\delta) K_t \leq F_t (K_t, N_t)$$

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- Can $N_t^* = 1$ be optimal? NO (same as decision problem, specification 1)
- How about $N_t^* = 0$? NO for a different reason:
  - $U_{1-N}(C, 1) < \infty$, but reasonable to assume $F_N(K, 0) = 0 \rightarrow N_t^* > 0$.
- So, labor supply choice always interior: $N_t^* \in (0, 1)$. (Different from decision problem)

Model Specification: Slight Change in Preferences?

$$\max_{\{C_t, N_t K_{t+1}\}} \quad \sum_{t=0}^{\infty} \beta^t \left[\frac{C_t^{1-\sigma}}{1-\sigma} - \psi \times \frac{{N_t}^{1+\gamma}}{1+\gamma} \right]$$

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- What changes between the two formulations?
- Can $N_t^* = 0$ be optimal? NO
  - Longer answer: yes but we rule it out by assuming $U_N(C, 0) < U_C(C, 0) F_N(K, 0)$ (i.e., what the first unit of labor produces is more valuable than disutility of first unit of labor).

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- How about $N_t^* = 1$ or ≥ 1? Yes (N_t has no natural upper bound here).

Model Specification: Cobb-Douglas Preferences

$$\max_{\{C_t, N_t K_{t+1}\}} \sum_{t=0}^{\infty} \beta^t \left[\frac{\left(C_t^{\alpha} (1 - N_t)^{1-\alpha}\right)^{1-\sigma}}{1-\sigma} \right]$$

$$s.t. \qquad C_t + K_{t+1} - (1-\delta) K_t \leq F_t (K_t, N_t) \qquad (\Lambda_t)$$

- What changes relative to power separable formulation?
- Non-separable utility:
  - More consistent with micro empirical evidence.
  - Can you get balanced growth?
- Again $N_t^* = 0$ possible, $N_t^* = 1$ is not.

Model Specification: GHH Preferences

$$\max_{\{C_t, N_t K_{t+1}\}} \quad \sum_{t=0}^{\infty} \beta^t \left[\frac{C_t - \psi N_t^{1+\gamma}}{1+\gamma} \right]^{1-\sigma}$$

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- Widely used [Greenwood et al., 1988] (GHH) preferences.

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- Generalizes quasi-linear utility that has no wealth/income effect by adding risk aversion.
  - No income effect on labor supply: $N = \frac{1}{\psi} w^{1/\gamma}$.

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- Note: POW and GHH have 3 distinct parameters, Cobb-Douglas has 2 $\rightarrow$ Less flexibility in setting RRA & Frisch elasticity separately in C-D.

General Dynamic Models with *Homogenous Solutions*

- u,F, and G are homogenous of degree 1 in their arguments.
- All constraints are linear.
- Result: Solution is homogenous of degree 1 in state variables.

$$\max \quad E\left\{\sum_{t=0}^{\infty}\beta^t u\left(c_t,\ell_t\right)\right\} \tag{1}$$

$$\text{s.t.} \quad c_t + x_{zt} + x_{ht} + x_{kt} \leq F\left(k_t, z_t, s_t\right) \tag{2}$$

$$z_t \leq M\left(n_{zt}, h_t, x_{zt}\right) \tag{3}$$

$$k_{t+1} \leq \left(1-\delta_k\right) k_t + x_{kt} \tag{4}$$

$$h_{t+1} \leq \left(1-\delta_h\right) h_t + G\left(n_{ht}, h_t, x_{ht}\right) \tag{5}$$

$$\ell_t + n_{ht} + n_{zt} \leq 1 \tag{6}$$

$$h_0 \text{ and } k_0 \text{ given}$$

Details

Here:

- $\{s_t\}$: first-order Markov chain with time-stationary transition probability function
- z_t is effective labor and n_{zt} is hours spent in the market working,
- x_{zt} is investment in effective labor (z_t)
- x_{kt} is investment in physical capital (k_t)
- x_{ht} is investment in human capital (h_t)
- n_{ht} is hours spent in augmenting human capital, and ℓ_t is leisure.

Assume that

$$u(c,\ell) = \begin{cases} v(\ell)\frac{c^{1-\gamma}}{1-\gamma} & \text{with } \gamma \neq 1, \text{ but } \gamma > 0 \\ log(c) + v(\ell) & \text{with } \gamma = 1 \end{cases}$$

Proposition 1
([Jones et al., 2000]) Assume that the utility function in (1) is homogeneous of degree $(1-\gamma)$ *in z (with n held fixed) and that the feasible set,* Γ*, is linearly homogeneous in* (h,k) *(with n and s held fixed) and that a solution exists for all* (h,k,s)*. Then the value function, V, for the problem above satisfies* $V(\lambda h,\lambda k,s)=\lambda^{(1-\gamma)}V(h,k,s)$*, for all* $\lambda>0$*. Moreover, the optimal choice of z is homogeneous of degree one (*$z^{\star}(\lambda h,\lambda k,s)=\lambda z^{\star}(h,k,s)$*) and the optimal choice of n is homogeneous of degree zero:*
$n^{\star}(\lambda h,\lambda k,s)=n^{\star}(h,k,s)$.

This proposition is the more general version of the Merton-Samuelson theorem and alike.

Changing Gears: What is Risk Aversion?

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- Alternatively, it will depend on how costly it is to prevent the risk from affecting consumption (or more generally, marginal utility).
- As we will see, sometimes the answer will have a simple relationship to the curvature, but oftentimes it will not.

What is Risk Aversion?

- Start with a *static* gamble as studied by [Pratt, 1964].
- Because the problem is static, there is no saving, so Pratt assumed the outcome of the gamble would be consumed immediately:
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- If the bet is declined, consumption is $\overline{c}$ minus the risk premium, π. So:

$$u(\overline{c} - \pi^a) = \sum_{i=1}^{n} p_i u(\overline{c} + \delta_i).$$

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- When the *risk is small*, use the Arrow-Pratt approximation.
- Basically, take the first-order Taylor approximation of the LHS, and the second-order approximation to the RHS (why?) to get:

$$
\begin{aligned}
u(\bar{c}) - \pi^a u'(\bar{c}) &= \sum_{i=1}^{n} p_i \left(u(\bar{c}) + \delta_i u'(\bar{c}) + \frac{1}{2}\delta_i^2 u''(\bar{c}) \right) \\
&= u(\bar{c}) \underbrace{\sum_{i=1}^{n} p_i}_{=1} + u'(\bar{c}) \underbrace{\sum_{i=1}^{n} p_i \delta_i}_{=0} + \frac{1}{2} u''(\bar{c}) \underbrace{\sum_{i=1}^{n} p_i \delta_i^2}_{=\text{var}(\delta_i)} \\
\pi^a u'(\bar{c}) &= -\frac{1}{2} u''(\bar{c}) \times \text{var}(\delta_i) \Rightarrow \\
\pi^a &= \underbrace{-\frac{u''(\bar{c})}{u'(\bar{c})}}_{\text{Absolute risk aversion}} \times \underbrace{\frac{1}{2}\text{var}(\delta_i)}_{\text{Amount of risk}}. \qquad (7)
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$$u(\bar{c}(1-\pi^r)) = \sum_{i=1}^{n} p_i u(\bar{c} \times (1+\delta_i)).$$

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- The coefficient of relative risk aversion:

$$RRA(c) = -\overline{c}\frac{u''(\overline{c})}{u'(\overline{c})} \tag{8}$$

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- So, in general, risk aversion will depend on the market structure and the *type of gamble* that is offered, so it can mean different things.
- In other words, it depends on what margins the agent has available to smooth consumption relative to bet's outcome (through borrowing/saving, labor supply, etc.)

Risk Aversion in a Dynamic Setting

In a dynamic model, individuals can "typically" use financial markets to smooth consumption, so we should think about wealth/income bets:

$$V(\omega(1-\pi^r)) = \sum_{i=1}^{n} p_i V(\omega(1+\delta_i)).$$

$$\pi^r = \underbrace{-\omega\frac{V''(\omega)}{V'(\omega)}}_{\text{Absolute risk aversion}} \times \underbrace{\frac{1}{2}\text{var}(\delta_i)}_{\text{Amount of risk}}. \tag{9}$$

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- **Result:** If (i) preferences are separable over time, and (ii) the market structure is such that (i.e., markets are complete) the envelope condition is $V'(\omega) = u'(c)\frac{\partial c}{\partial \omega}$, then:

$$-\omega\frac{V''(\omega)}{V'(\omega)} = -\bar{c}\frac{u''(\bar{c})}{u'(\bar{c})},$$

where we used Euler's theorem that $\frac{\partial c}{\partial \omega}\omega = c$.

Risk Aversion in a Dynamic Setting

- This explanation also makes it clear that this result is more special and limited than it looks.
- Because we know that in many models the marginal utility of consumption is not equated across dates and states, most notably when markets are incomplete—which is most of the models this course covers!
- In such cases, immediately consuming the outcome of the bet cannot be any greater than finding the state with the highest marginal utility and consuming in that state.

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- In such cases, immediately consuming the outcome of the bet cannot be any greater than finding the state with the highest marginal utility and consuming in that state.
- So wealth will have (weakly) higher marginal utility than current consumption yielding an inequality:

$$-\omega \frac{V''(\omega)}{V'(\omega)} \geq -\bar{c}\frac{u''(\bar{c})}{u'(\bar{c})} = \alpha. \tag{10}$$

Non-Separable Utility

- A second case of interest is when preferences are time-non-separable, e.g., Epstein-Zin preferences or habit formation.
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- If it is literally financial wealth, risk aversion may be zero or negative as measured by eq. (10), since w could be zero or negative.
- If we think that it should include labor income, so it is cash-on-hand, then how do we discount future earnings? In general, the formula above is not very useful in incomplete markets models as a measure because of these difficulties.

Dynamic Setting: Risk Aversion with Labor Supply

- Consider a one-shot wealth gamble of size A_t:

$$a_{t+1} = (1 + r_t)a_t + w_t\ell_t - c_t + A_t\sigma\varepsilon_{t+1}.$$

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- Pay risk premium $A_t\mu$ to avoid gamble. RRA is defined as

$$lim_{\sigma\to 0}\frac{2\mu(\sigma)}{\sigma^2} = -\frac{A_t\mathbb{E}_t V_{11}(a^*_{t+1};\theta_{t+1})}{\mathbb{E}_t V_1(a^*_{t+1};\theta_{t+1})}$$

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- Typically, we define A_t to be a fraction of household's wealth at time t.
- Question: what is a sensible definition of wealth in a dynamic model with labor supply?
  - (The answer matters for many questions beyond the current context)

Risk Aversion with Labor Supply, Cont'd

- One definition that makes sense is the (properly) discounted value of future resources, either based on future consumption alone:

$$A_t \equiv (1 + r_t)^{-1} \mathbb{E}_t \sum_{\tau=t}^{\infty} m_{t,\tau} c_\tau^*$$

where $m_{t,\tau} \equiv \beta^{\tau-t} u_1(c_\tau^*, \ell_\tau^*)/u_1(c_t^*, \ell_t^*)$ is individual's stochastic discount factor; or including future values of leisure time:

$$\tilde{A}_t \equiv (1 + r_t)^{-1} \mathbb{E}_t \sum_{\tau=t}^{\infty} m_{t,\tau} \left(c_\tau^* + (\bar{\ell} - \ell_\tau^*) \right).$$

- Power separable specification:

$$u(c_t, \ell_t) = \frac{c_t^{1-\gamma}}{1-\gamma} - \eta \frac{\ell^{1+\chi}}{1+\chi},$$

with $\gamma, \eta, \chi > 0$. We have

$$RRA^c = \frac{\gamma}{1 + \frac{\gamma}{\chi}\frac{w\ell}{c}} \approx \frac{1}{\frac{1}{\gamma} + \frac{1}{\chi}},$$

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4 Cases: Case 1

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- For example, if $\gamma = 2$ and $\chi = 1$, RRA^c is $\gamma/3$.
- More interesting: if $\chi = 0$, so linear utility in labor, then $RRA^c = 0$!
- BUT timing is also crucial! [Boldrin et al., 2001] consider this case and study its asset pricing implications.
  - If agent chooses ℓ *after* observing shock and consumes afterwards, $RRA^c = 0$, there is no risk premium.
  - If instead, agent chooses ℓ first, then observes shock and consumes, risk aversion and risk premium can be very high.

Case 2

2. Cobb-Douglas specification:

$$u(c_t, \ell_t) = \frac{(c_t^{1-\chi}(1-\ell_t)^{\chi})^{1-\gamma}}{1-\gamma},$$

with $\chi \in (0,1)$, we have

$$RRA^{cl} = \gamma,$$

since consumption and leisure act as a single composite commodity subject to the same risk aversion.

Cases 3 and 4

3. King-Plosser-Rebelo (KPR) preferences:

$$u(c_t, \ell_t) = \frac{c_t^{1-\chi}(1-\ell_t)^{\chi(1-\gamma)}}{1-\gamma},$$

with $\gamma, \chi > 0$, and $\chi(1-\gamma) < \gamma$ for concavity. Here we have

$$RRA^{cl} = \gamma + \chi(\gamma - 1).$$

- Note that $RRA^{cl} > \gamma$ as long as $\gamma > 1 \Rightarrow$ leisure margin does not always reduce risk aversion, it can also increase as in this example!

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$$u(c_t, \ell_t) = \left(c - \psi\frac{\ell^{1+\chi}}{1+\chi}\right)^{1-\gamma} / (1-\gamma)$$

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- See [Boldrin et al., 1997] and [Swanson, 2012] for more details.

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- Taylor's Theorem: If f^n exists for all n in an open interval I containing a then for each positive integer n and for each x in I,

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{2}(x-a)^3 + \ldots$$
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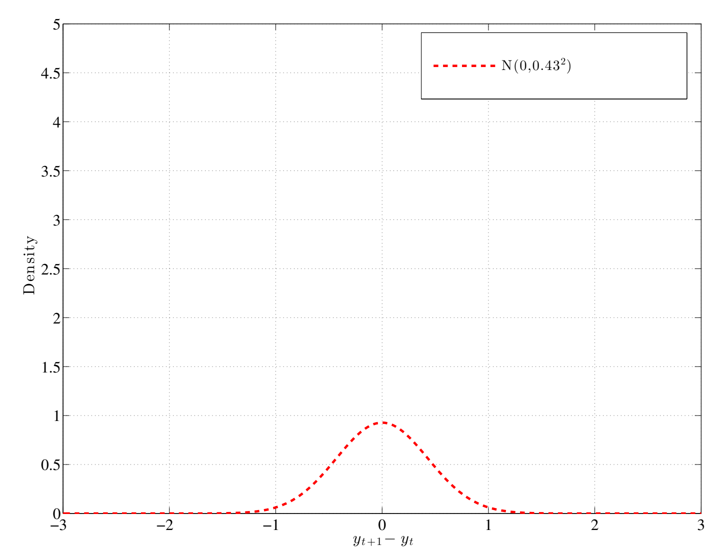
- Key question: what happens to $f^{(n+1)}(c)$ as n grows? For a polynomial function (n positive integer) it goes to zero. But for a function with a negative exponent, it will get larger if $c < 1$.

2. Long-Tailed Risk: Income Shocks



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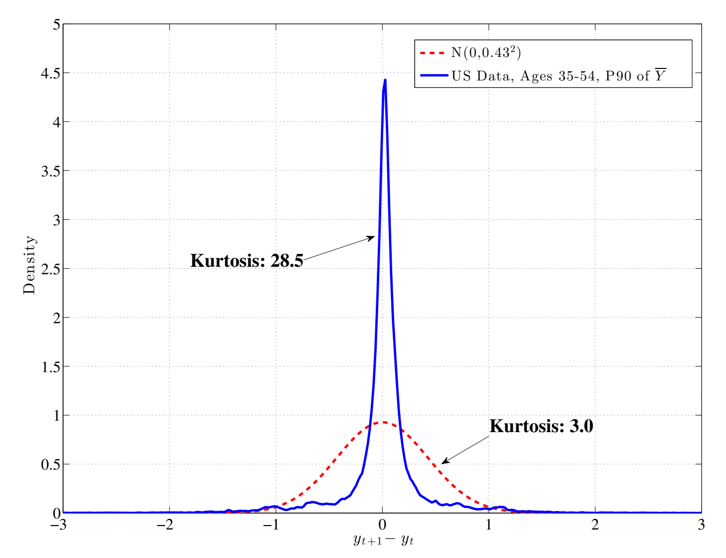
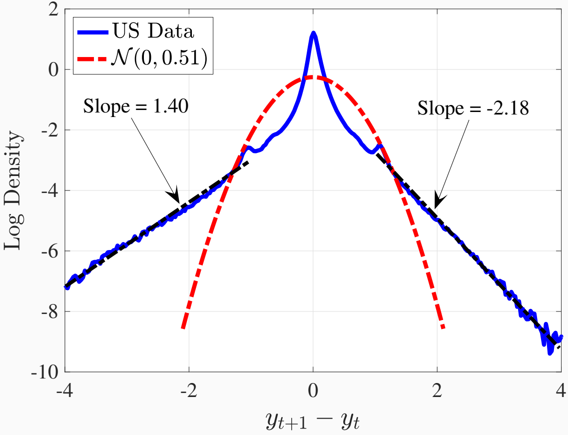


Plot Log Density to See Tails Better: They are Double-Pareto



- $3\sigma+$ shocks 9X more likely, $4\sigma+$ shocks 40X more likely than a Gaussian.

Aversion to Higher-Order Risk

$$U(c(1-\pi)) = \mathbb{E}\left(U(c(1+\tilde{\delta})\right)$$

- Take fourth-order Taylor approximation to RHS: $U(c) - U'(c)c\pi =$

$$= \mathbb{E}\left(U(c) + U'(c)c\tilde{\delta} + \frac{1}{2}U''(c)c^2\tilde{\delta}^2 + \frac{1}{6}U'''(c)c^3\tilde{\delta}^3 + \frac{1}{24}U''''(c)c^4\tilde{\delta}^4\right).$$

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- Second term on the RHS is zero when $\mathbb{E}(\tilde{\delta}) = 0$, so rearranging yields:

$$\pi = -\frac{1}{2}\frac{u''(c)c}{u'(c)} \times m_2 - \frac{1}{6}\frac{u'''(c)c^2}{u'(c)} \times m_3 + \frac{1}{24}\frac{u''''(c)c^3}{u'(c)} \times m_4, \qquad (11)$$

where m_n denotes the n^{th} central moment of $\tilde{\delta}$.

Higher Order Risk Aversion

- To convert these into statistics that are more familiar, write $m_2 = \sigma_\delta^2$, $m_3 = s_\delta \times \sigma_\delta^3$, where s_δ is the skewness coefficient, and $m_4 = k_\delta \times \sigma_\delta^4$, where k_δ is kurtosis.
- With this notation, and assuming a CRRA utility function with curvature θ, we get:

$$\pi^* = \frac{\theta}{2} \times \sigma_\delta^2 \underbrace{-\frac{(\theta+1)\theta}{6} \times s_\delta \times \sigma_\delta^3}_{\text{Negative skewness aversion}} + \underbrace{\frac{(\theta+2)(\theta+1)\theta}{24} \times k_\delta \times \sigma_\delta^4}_{\text{Kurtosis aversion}}$$

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- which can also be written as:

$$\pi^* = \frac{\theta}{2} \times \sigma_\delta^2 \times \left[1 + \underbrace{\frac{1}{3}(\theta+1)\left(-s_\delta \times \sigma_\delta + \frac{1}{4}(\theta+2) k_\delta \times \sigma_\delta^2 \right)}_{\text{Aversion to Higher-OrderRisk}} \right].$$

Risk Premium: Skewness and Kurtosis

Let $\tilde{\delta}$ be a static gamble. And π is the risk premium to avoid it:

$$U(c \times (1 - \pi)) = \mathbb{E}\left[U(c \times (1 + \tilde{\delta}))\right].$$

	Risk Premium (π)	
Gamble:		
Mean	0	0
Standard Deviation	0.10	0.10
Skewness	0	-2
Excess Kurtosis	0	27
Premium	4.88%	22.15%

Risk Premium: Skewness and Kurtosis

Let $\tilde{\delta}$ be a static gamble. And π is the risk premium to avoid it:

$$U(c \times (1 - \pi)) = \mathbb{E}\left[U(c \times (1 + \tilde{\delta}))\right].$$

	Risk Premium (π)	
Gamble:	$\tilde{\delta}^A$	$\tilde{\delta}^B$
Mean	0.0	0
Standard Deviation	0.10	0.10
Skewness	0.0	−2
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- ∴ Higher-order risk can matter greatly for economic questions!

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