A Appendix: Estimates of the HIP Model in the Literature

Table 4 presents the estimates of the HIP model from the U.S. data in the previous literature. As can be seen here, the estimates of σ_β^2 range from 0.00018 in Lillard and Weiss (1979) to 0.00041 in Haider (2001). The former paper estimates a separate income process for each finely defined occupation category (such as chemists, psychologists, etc.), which could be partly responsible for the smaller estimate of profile heterogeneity. However, all the estimates of σ_β^2 are statistically significant, and the latter two papers' point estimates are rather close to each other. Baker also reports estimates as high as 0.00082; his lowest estimate is 0.00031. Second, the persistence parameter in these studies is around 0.6 to 0.7, indicating significantly lower persistence than a unit root.

B Appendix: Computational Algorithm

This appendix describes the algorithm used to solve the consumption-savings problem described in Section 3. The first point to observe is that since the value function does not explicitly depend on the type of individual we need to solve for only one value function for all individuals. The true type only determines the probability distribution of income (induced by the probability distributions of η and ε for a given $\left(\alpha^i, \beta^i\right)$), which then determines the probability distribution of the belief vector, $\widehat{\mathbf{S}}^i_{t|t-1}$, for a given agent. In turn, this determines which region of the state space will be most visited for a given individual. To solve the model for a large number of types, we need to get a good approximation of the value function for the union of the supports for these different types, which is the challenging part.

We first describe the algorithm for $\lambda = 0$ so that all individuals begin life with the same prior information. A slight modification then will solve the model for different λ values. The critical part of the algorithm is the construction of a convenient grid over which the dynamic problem is solved. Once this is accomplished, solving the model is straightforward.

Step 0: Grid construction

1. Draw I types $\left\{\left(\alpha^i, \beta^i\right), i = 1, .., I\right\}$ from a Normal distribution with second moments $(\sigma_\alpha^2, \sigma_\beta^2, \sigma_{\alpha\beta})$ reported in Table 1. In the baseline case, we chose $I = 1000$.

2. For each i, simulate J income paths $\left\{\widetilde{y}_t^{ij},\ t = 1, .., T;\ j = 1, .., J\right\}$ using equation (2) to obtain an empirical approximation to the distribution of $\widetilde{y}_t^i$. We chose $J = 100$.

3. For each of the $N \equiv I \times J$ income paths, use equation (3) to obtain a sequence of $\widehat{\mathbf{S}}^{ij}_{t|t-1}$ for $t = 1, .., T$. Thus, for each t, we have $N = 100,000$ points distributed over the 3-dimensional space of beliefs, $(\widehat{\alpha}^i_{t|t-1}, \widehat{\beta}^i_{t|t-1}, \widehat{z}^i_{t|t-1})$. Instead of choosing independent grids in $\widehat{\alpha}^i_{t|t-1}$, , $\widehat{\beta}^i_{t|t-1}$, and $\widehat{z}_{t|t-1}$ directions and taking the Cartesian product of these intervals, we directly choose points in this 3-dimensional space as follows. We divide the space $[\widehat{\alpha}_{\min}, \widehat{\alpha}_{\max}] \times [\widehat{\beta}_{\min}, \widehat{\beta}_{\max}] \times [\widehat{z}_{\min}, \widehat{z}_{\max}]$ (with appropriately chosen lower and upper bounds) into cubes by taking 21 points in each direction (and get $20 \times 20 \times 20$ cubes). For every t, if there are any points (among the 100,000 realizations of $\widehat{\mathbf{S}}^{ij}_{t|t-1}$) that fall into a cube, we assign a grid point to the center of that cube (and eliminate all empty cubes). This procedure picks a subset of the 3-dimensional space that contains state points that have a non-negligible probability of being realized when we simulate the model. (It is important to emphasize that we do *not* do this for efficiency reasons. Our experience is that attempts at solving for the value function over a Cartesian state space runs into a number of difficulties, and this is one approach we found to work.) We enumerate these triplets $\{\widetilde{\mathbf{S}}^q_t = (\widehat{\alpha}, \widehat{\beta}, \widehat{z})^q, q = 1, .., Q_t\}$, where Q_t is the total number of non-empty cubes and hence grid points at age t (From this point on, we drop the reference to t and describe the grid construction for a given age. The same procedure is repeated for each t.)

4. The grid for $\widetilde{y}^{ij}$ needs to be consistent with the probability distribution implied by the type of individual, otherwise one runs into a number of problems.[27] However, since $\left(\alpha^i, \beta^i\right)$ is not a state variable it is

[27] For example, if we attempt to solve the dynamic problem with a $\widetilde{y}^h_{it}$ that is much larger than what would be

not possible to literally have the grid for $\widetilde{y}^{ij}$ depend on the type. Instead then, we choose a different grid for each possible belief vector, $\widetilde{\mathbf{S}}^q$, defined as $\mathbf{y}^q_{grid} = [y^q_{\min}, y^q_{\max}]$, where the bounds are defined as $\exp(H\widetilde{\mathbf{S}}^q \pm 3\sigma(\widetilde{y}^q))$; $H\widetilde{\mathbf{S}}^q$ is the mean income and $\sigma(\widetilde{y}^q)$ is the standard deviation given in equation (5). In other words, these bounds define a three standard deviation confidence interval for income through equation (5) given beliefs $\widetilde{\mathbf{S}}^q$. We take 8 equally spaced points for each income grid. (Using 20 points did not make a noticeable difference in results.) We repeat the same steps for each t.

5. Unlike the other 4 state variables, wealth does not affect and is not affected by the learning process. Thus, we take a fixed wealth grid—that is, one that does not depend on beliefs or income—with 12 points more densely spaced near the borrowing constraint. (Using 20 points did not make a noticeable difference in results.) At a given age, the final grid is the Cartesian product of this wealth grid and the (4-dimensional) grid $(\mathbf{y}^q_{grid}, \widetilde{\mathbf{S}}^q)$. So the problem is solved on $(12 \times 8 \times Q_t)$ grid points, where Q_t ranges from 240 to 1100 over the lifecycle and averages 830.

Step 1: Solving the dynamic problem

1. The dynamic problem is solved using the Bellman equation approach. We solve the problem for each point on the random grid at age t.

2. The non-Cartesian structure of the state space rules out a number of multi-dimensional interpolation methods such as splines, Chebyshev polynomials that typically require Cartesian grids in more than one dimension. Instead, we approximate the value function with a combination of polynomial functions (up to the 4th power) and other functions (such as logs and fractional powers) of the state variables including various interaction terms between them (a total of 162 terms used in the baseline model). After solving the Bellman equation at age t, we regress the values of the value function at the grid points on these functions of the state variables. These coefficients are then used for the interpolations necessary to evaluate the expectation when solving the period $t-1$ problem.

3. After the model is solved, we simulate the decision rules for a large number of individuals. For simplicity we used the same I types drawn above and the N simulated income paths to obtain consumption-savings paths.

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implied by the individual's (α^i, β^i), the Bayesian updating results in next period's beliefs that are substantially away from next period's grid for $\widetilde{\mathbf{S}}^q_{t+1}$, because the latter is constructed based on income realizations that are going to be observed in the actual solution. As a result, one needs to extrapolate next period's value function which often yields extremely inaccurate results (despite the fact that these far-off points have low probability). Considering a $\widetilde{y}^h_{it}$ that is much smaller than what is consistent with the type, results in similar problems as well as creating further problems with infeasible borrowing constraints.